Bifurcations of Hamiltonian Periodic Orbits

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Overview

Well-known Classical Fact from 70s

There are 8 types of *generic* bifurcation of Hamiltonian periodic orbits.

Contents

- Dynamical Systems and Bifurcations
- Elliptic and Hyperbolic Orbits
- 8 Types of Generic Bifurcations

Main references : [Mey70], [AM78], [MHO13].

Dynamical Systems and Bifurcations

Toy Example : Critical Points of a Function

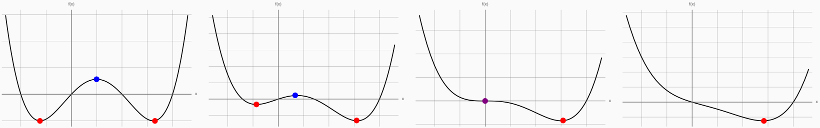


Figure 1: Graphs of $f_t(x) = x(x^2 + t)(x - 1)$ for varying t.

- $t < 0$, two minima (●) and one maximum. (●)
- $t = 0$, ● and ● meets at the inflection point. (●)
- $t > 0$, ● vanishes and one minimum survives. (●)

Idea. Similar thing happens for periodic orbits.

Example : Van der Pol Oscillator

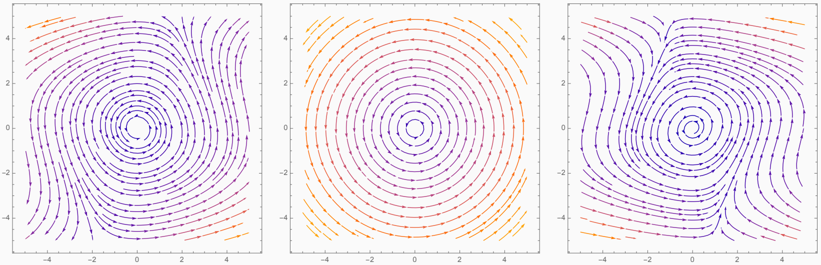


Figure 2: Phase portraits of Van der Pol oscillator for varying μ.

Van der Pol oscillator is given by $\dot{x} = \mu(1 - y^2)x - y$, $\dot{y} = x$.

0 is a fixed point, and its property changes as μ varies.

Also, at $\mu = 0$, a family of periodic orbits emerge.

Dynamical System

According to [Sma67], a **dynamical system** consists of following data.

- A space W (we use smooth manifolds)
- A group action $G \to \mathrm{End}(W)$ (we use diffeomorphisms)

If $G = \mathbb{Z} = \langle F \rangle$, we say the system is **discrete**.

Interesting objects

- **Fixed point**: $x \in W$ such that $F(x) = x$.
- **Periodic point**: $x \in W$ such that $F^N(x) = x$ for some N.
- **Invariant subspace**: $S \subset W$ such that $F(S) \subset S$.
- Non-wandering points, recurrent points, etc.

Continuous Dynamical System

If $G = \mathbb{R} = \{F_t\}$, we say the system is **continuous**.

A vector field $\frac{d}{dt}F_t|_{t=0} = X$ consists equivalent information with F_t, which is called the infinitesimal generator of F_t.

Example. The system determined by an ODE $\dot{x} = X(x)$.

Interesting objects

- **Critical points**: $x \in W$ such that $F_t(x) = x$ for any t. (or $X_x = 0$.)
- **Periodic orbits**: the orbit $\{F_t(x) : t \in \mathbb{R}\}$ for $x \in W$ such that $F_\tau(x) = x$ for some $\tau > 0$.
- **Invariant subsets** $S \subset W$ such that $F_t(S) \subset S$ for any t (or $t > 0$).

Hamiltonian Dynamical System

Let (W, ω) be a symplectic manifold, $H : W \to \mathbb{R}$.

H defines a **Hamiltonian vector field** X_H via relation

$$i_{X_H}\omega = \omega(X_H, -) = -dH.$$

We call (W, ω, H) a **Hamiltonian dynamical system**.

- X_H preserves ω, i.e. $\mathcal{L}_{X_H}\omega = 0$.
- X_H preserves H, i.e. X_H flows on $H^{-1}(c)$ for each c.

The first example is the geodesic flow on a Riemannian manifold (M, g),

$$H_g(q, p) = \frac{|p|_g^2}{2}$$

which is defined on T^*M.

Example : Classical Mechanics

Consider the mechanical system in $(T^*\mathbb{R}^n, \sum dp_i \wedge dq_i)$ with energy

$$H(q,p) = \frac{|p|^2}{2} + V(q).$$

Corresponding Hamiltonian vector field is

$$X_H = p\frac{\partial}{\partial q} - \nabla V \frac{\partial}{\partial p},$$

which can be translated into

$$\dot{q} = p, \quad \dot{p} = \ddot{q} = -\nabla V.$$

This is Newton's 2nd law, which governs the classical mechanics.

Bifurcation

Two vector fields X, Y on W are **topologically conjugate** if there exists a homeomorphism $F : W \to W$ which carries the oriented orbits of X to the ones of Y.

It means that the dynamics of X and Y are essentially the same.

Consider a family of vector fields X_t on W, parametrized by a set I.

A point $s \in I$ is called

- a **robust point** if there exists a neighborhood U of s in I such that if $t \in U$, X_s and X_t are topologically conjugate.
- a **bifurcation point** if not.

We will only consider 1-dimensional bifurcation, i.e. I is 1-manifold.

Bifurcation

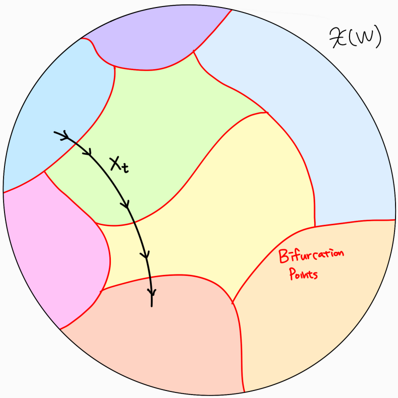


Figure 3: Very schematic picture of a bifurcation.

Bifurcation of Our Interest

Setting. Hamiltonian vector field X_H on 4-dimensional (W, ω).

Consider a periodic orbit γ or a critical point x at $H = c$.

What happens if we change the energy level?

1. A qualitative change of x, or γ and its multiple covers.
2. An appearance or disappearance of another orbits near γ or x.
3. Properties of the new orbit. (elliptic / hyperbolic)

Elliptic and Hyperbolic Orbits

Elliptic and Hyperbolic Critical Points

Let x be a critical point of X_H on W.

The **characteristic exponents** of x are the eigenvalues of $dX_H(x)$.

Since $dX_H(x) \in \mathfrak{sp}(2n)$, these come with pairs, $\pm z, \pm \bar{z} \in CE(x)$.

Let $CE(x) = \{\alpha_j\}$ be characteristic exponents of x.

- If $0 \neq \alpha_j \in i\mathbb{R}$ for all j, call x **elliptic**.
- If $\alpha_j \notin i\mathbb{R}$ for all j, call x **hyperbolic**.

In 4-dimensional case, we have following non-degenerate $(\alpha \neq 0)$ cases:

- Elliptic: $CE(x) = \{\pm i\alpha, \pm i\beta\}$ for $0 \neq \alpha, \beta \in \mathbb{R}$,
- Saddle-center: $CE(x) = \{\pm i\alpha, \pm r\}$ for $0 \neq \alpha, r \in \mathbb{R}$,
- Hyperbolic: $CE(x) = \{\pm z, \pm \bar{z}\}$ for $0 \neq z \in \mathbb{C}$,
  or $CE(x) = \{\pm r, \pm t\}$ for $0 \neq r, t \in \mathbb{R}$.

Elliptic and Hyperbolic Critical Points

Let α be a characteristic exponent of x, $\lambda = e^{\alpha}$ and v be eigenvector.

Consider the linearized flow in direction v at a critical points.

- If $|\lambda| = 1$, the flow rotates around x. (**center**)
- If $|\lambda| > 1$, the flow is expanding and tends away from x. (**unstable**)
- If $|\lambda| < 1$, the flow is contracting and tends closer to x. (**stable**)

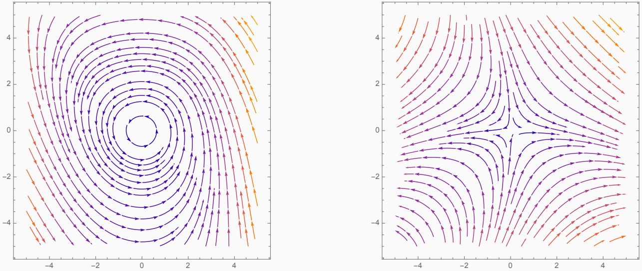
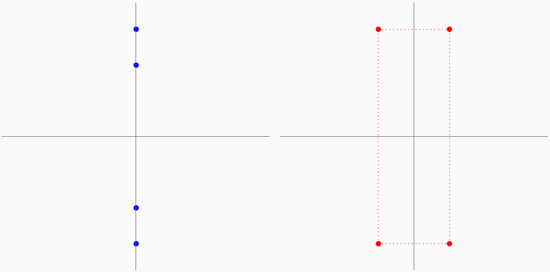


Figure 4: Elliptic and hyperbolic fixed points

Rigidity of Critical Points of Symplectomorphism



(a) Elliptic Exponents **(b)** Hyperbolic Exponents

Consider the elliptic critical point x with $CE(x) = \{\pm i\alpha, \pm i\beta\}$.

Under perturbation, α and β change continuously, but locked in $i\mathbb{R}$ unless $\alpha = \beta$ at some point. So it cannot suddenly become hyperbolic.

Poincaré Section

Theorem (Poincaré Section and Return Map)

Let γ be a periodic orbit of X in W and $x \in \gamma$.

1. *There exists a codimension 1 submanifold $S \subset W$ such that $x \in S$ and is transversal to X.*
2. *Define a map $\Psi : S \to W$ by*

$$\Psi(y) = Fl^X_{\tau(y)}(y)$$

where $\tau(y) = \min\{t > 0 \,:\, Fl^X_t(y) \in S\}$, then this is well-defined and unique up to conjugation.

We call S **Poincaré section**, and Ψ **return map** or **Poincaré map**.

Poincaré Section

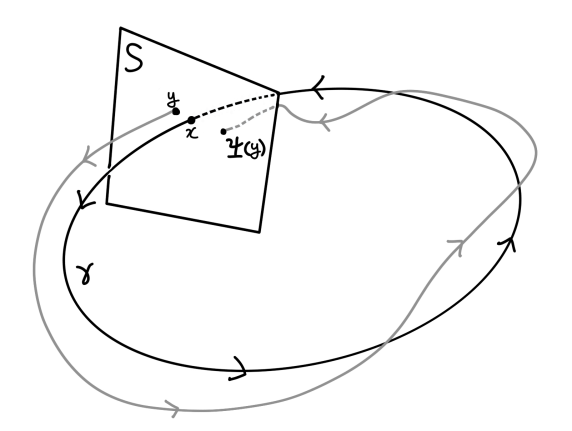


Figure 6: A Poincaré section and the return map

Hamiltonian Version of Poincaré Section

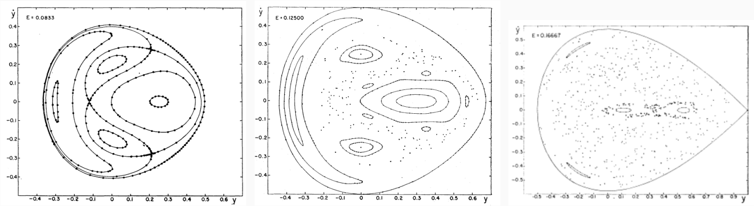
Proposition

Let $X = X_H$ be a Hamiltonian vector field and γ be a periodic orbit with regular energy c_0. There exists a Poincaré section Y such that

1. *$Y \subset W$ is a contact submanifold.*
2. *$S_c = Y \cap H^{-1}(c)$ for c close to c_0 is a symplectic submanifold.*
3. *The return map is a symplectomorphism.*

$2n$-dimensional W	$\Leftrightarrow$	$(2n-2)$-dimensional S
Flow of vector field X_H	$\Leftrightarrow$	a symplectomorphism Ψ
1-periodic orbit of X_H on W	$\Leftrightarrow$	a fixed point of Ψ on S
k-periodic orbit of X_H	$\Leftrightarrow$	k-pair of k-periodic points of Ψ.

Aside : A Standard Picture the Poincaré Section



(a) $c = 0.0833$ **(b)** $c = 0.125$ **(c)** $c = 0.167$

Figure 7: [HH64] Poincaré sections of the Hénon-Heiles system

1. Take an initial condition x_0 and compute its orbit γ.
2. Mark the intersections x_i's of γ and S.
3. Draw a curve (if possible) between x_i's.
4. Repeat for different initial conditions.

Characteristic Multipliers of Periodic Orbits

Let γ be a periodic orbit, S be a Poincaré section and $\gamma \cap S = \{x\}$.

The **characteristic multipliers** of γ are the eigenvalues of $d\Psi_x$.

Since $d\Psi_x \in Sp(2n)$, the multipliers come with pairs, $\lambda^{\pm 1}$, $\bar{\lambda}^{\pm 1}$.

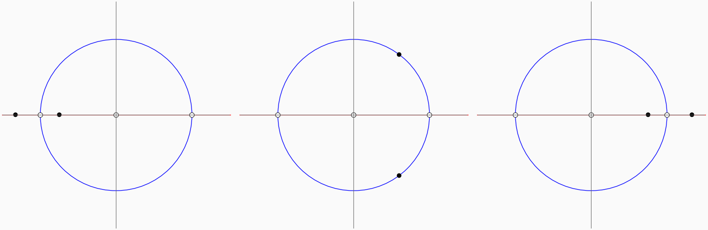
If $\dim S = 2$, which is of our interest, it simplifies by $\lambda \in S^1 \cup \mathbb{R} \setminus \{0\}$.

- If $\lambda_\gamma \in \mathbb{R} \setminus \{\pm 1\}$, call γ **hyperbolic**.

  We say γ is **positive** or **negative** depending on the sign of λ_γ.
- If $\lambda_\gamma \in S^1 \setminus \{\pm 1\}$, call γ **elliptic**.

We will abuse the notation by $\lambda_\gamma = \lambda_x$ sometimes.

Characteristic Multipliers



(a) $\lambda_\gamma < 0$ **(b)** $|\lambda_\gamma| = 1$ **(c)** $\lambda_\gamma > 0$

As in the critical points, the transition between elliptic and hyperbolic periodic orbits can be done only through ± 1.

Aside: Birkhoff-Lewis Theorem

Proposition ([BL34] Birkhoff-Lewis)

Let $\Psi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a symplectomorphism with an elliptic fixed point $\Psi(0) = 0$. Then generically, in every neighborhood of 0,

1. *there exists an invariant circle in U without periodic points.*
2. *such invariant circles have positive measure.*
3. *for any integer k, there exists a k-periodic point in U.*

It means the elliptic periodic orbits can be detected from nearby orbits, while the hyperbolic ones cannot be.

In this sense, the hyperbolic periodic orbits are *qualitatively invisible*.

Note. The *generic* condition here is *elementary twist condition*, explained in Appendix 1.

Aside : Vague Attractor of Kolmogorov

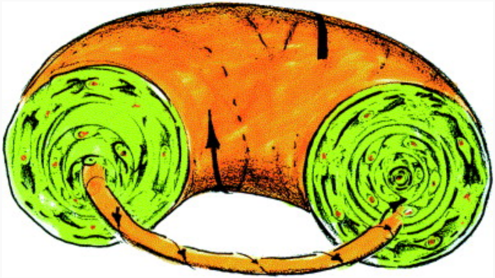


Figure 9: [El 03] VAK

- We have invariant tori (of positive measure) around γ.
- Between tori, there are periodic orbits. (red dots)

We say such configuration a **vague attractor of Kolmogorov**, or **VAK**.

8 Types of Bifurcations

Orbit Cylinder

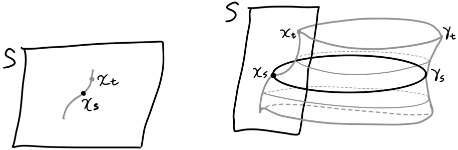
Let γ be a periodic orbit, S be a Poincaré section at x.

We consider a family of symplectomorphisms $\varphi : S \times I \to S$.

If x is a fixed point of φ_s, we say (x, s) is a fixed point of φ.

Lemma (Orbit Cylinder)

If $\lambda_x \neq 1$, there exists neighborhoods $x \in U \subset S$ and $s \in V \subset I$ and a smooth map $P : V \to U$ such that $P(s) = x$ and $\{(P(t), t) : t \in V\}$ consists of fixed points.



Two Types of Orbits Cylinders

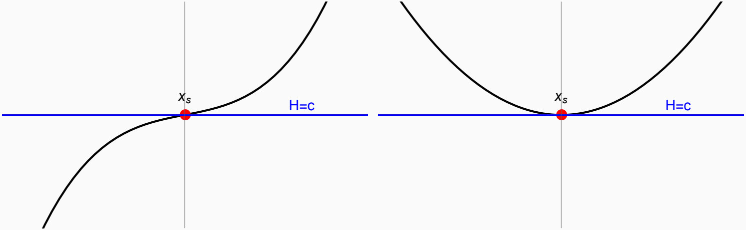


Figure 10: Two possible orbit cylinders

The *height parameter* of the orbit cylinder can be chosen as the energy.

There are two possibilities; the orbit cylinder might be transversal or tangent to the energy hyperplane.

How to Classify?

Let $\dim W = 4$, so the Poincaré section S is 2-dimensional.

Critical Points

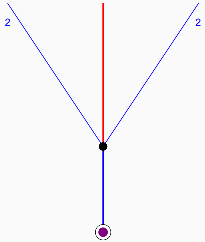
- For critical points, $CE(x) = \{\pm z, \pm \bar{z}\}$ in general.
- When there is $i\alpha \in CE(x)$, Lyapunov theorem can be applied. (I)
- If $CE(x_s) = \{\pm i\alpha_t, \pm i\beta_t\}$ and $l\alpha_s = k\beta_s$, for some s and $k, l \in \mathbb{Z}$, the Lyapunov assumption is violated. (VII, VIII)

Periodic Orbits

- For periodic orbits, if $\lambda_\gamma \neq 1$, there is an orbit cylinder.
- If $\lambda_\gamma^k = 1$, k-th cover of the orbit bifurcates. (II – V)
- We'll analyze by looking at the fixed points of the Poincaré section.

Bifurcation Diagram

We'll see this kind of diagrams.



- Vertical axis is the energy.
- • is the bifurcation point.
- ⊙ is the critical point of H.
- Each line is a family of periodic points of the return map in Poincaré section.
- Thick line is a family of fixed points, or 1-periodic orbits.
- Thin line is a family of k-periodic orbit, where the period is written aside the line. (2 here.)
- —, ⊙ are elliptic, —, ⊙ are hyperbolic, and ⊙ is saddle.

I. Burst / Reincarnation

Critical points with $i\alpha \in CE(x)$

Lyapunov's Theorem

Let x be a critical point of X_H. x is called $\mathcal{H}$-**elementary** if

1. 0 is not a characteristic exponent.
2. If $i\alpha$ is a characteristic exponent, $i\alpha$ has multiplicity 1.
3. If $i\alpha, i\beta$ are characteristic exponents with $\alpha \neq \pm\beta$, they are $\mathbb{Z}$-linearly independent.

Theorem ([Buc70])

For generic Hamiltonian H, every critical point of X_H is $\mathcal{H}$-elementary.

Lyapunov's Theorem

Theorem (Lyapunov)

Let x be an $\mathcal{H}$-elementary critical point of X_H. If $\pm i\beta$ is a characteristic exponent for $\beta \in \mathbb{R}$, there exists a 2-dimensional submanifold C_β such that

1. $x \in C_\beta$.
2. C_β *is* X_H*-invariant submanifold.*
3. $T_x C_\beta$ *is an eigenspace corresponds to* $\pm i\beta$.
4. C_β *is a union of periodic orbits* γ_r *which converges to* x *as* $r \to 0$.
5. *The period of* γ_r *converges to* $2\pi/\beta$ *as* $r \to 0$.

Simply, there exist families of periodic orbits emerging from x for each elliptic exponent.

Lyapunov Orbits

Example. Lyapunov orbits of the restricted three-body problem, emerges from the Lagrange points.

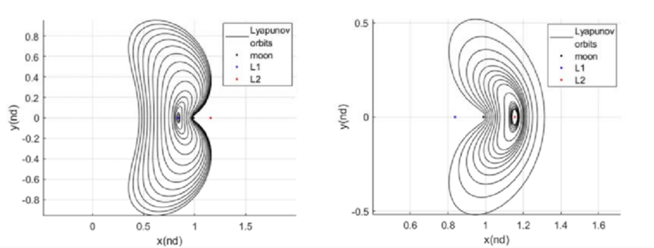


Figure 11: [ZWZZ20] Lyapunov orbits of RTBP, associated to L_1 and L_2.

These Lagrange points have only one pair of elliptic exponents, so there is only one family of orbits.

Type I. Phantom Burst

Target. Critical point x with pure imaginary characteristic exponents.

Case 1. $\{\pm i\alpha, \pm r\}$ with $\alpha, r \in \mathbb{R}$. (saddle-center)

Figure 12: Phantom burst

- By Lyapunov's theorem, we have a family of orbits near x
- The characteristic multipliers of the Lyapunov orbits near x is eventually real, so the orbits are hyperbolic.
- This hyperbolic family cannot be observed in the phase portrait, so we call this a **phantom burst**.

Type I. Stable Burst and Reincarnation

Case 2. $\{\pm i\alpha, \pm i\beta\}$ with $\mathbb{Q}$-independent $\alpha, \beta \in \mathbb{R}$. (pure center)

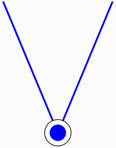


Figure 13: Stable burst

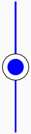


Figure 14: Reincarnation

- By Lyapunov's theorem, we have two families near $\odot$.
- The characteristic multipliers are unimodular, so both orbits are elliptic.
- If two orbit cylinders are on the same side respect to the energy hyperplane, we observe two families of orbits burst from the critical point. (**Stable burst**)
- Otherwise, we observe a family of orbit absorbed into the critical point and burst out again. (**Reincarnation**)

II. Creation / Annihilation

Periodic orbit such that $\lambda_\gamma = 1$

Extremal Fixed Point

Now we only focus on the cases $\lambda_x^k = 1$ for some k.

We will see $\lambda_x = 1$ case in detail, and other cases briefly.

We call (x, s) is **extremal fixed point** if locally we can write

$$\varphi_t(q,p) = \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + t \begin{pmatrix} \gamma \\ \delta \end{pmatrix} + \begin{pmatrix} \cdots \\ \beta q^2 + \cdots \end{pmatrix} + \cdots$$

with $\alpha = \pm 1$, $\beta, \delta \neq 0$.

Note. [Mey70] defines this in terms of generating functions.

Meyer's Theorem - Extremal Fixed Point

Theorem ([Mey70])

Let (x, s) be an extremal fixed point. Then,

1. *There exists a family of fixed points x_t.*
2. *x divides x_t into two arcs: one entirely consists of hyperbolic fixed points and the other consists of elliptic fixed points.*
3. *In any coordinate system, s is a nondegenerate maximum or minimum of the family.*

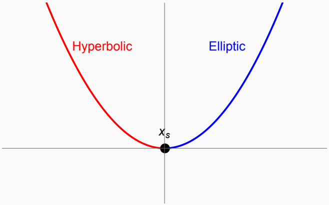


Figure 15: Meyer's theorem for extremal fixed point

Outline of the Proof

We need to solve, for $\alpha = \pm 1$, $\beta, \delta \neq 0$,

$$0 = Q - q = \alpha p + \gamma t + \cdots ,$$
$$0 = P - p = \delta t + \beta q^2 + \cdots .$$

There exists a solution $(q, p, t) = (\tau, p(\tau), t(\tau))$, since $\alpha, \delta \neq 0$, which are

$$p(\tau) = O(\tau^2), \quad t(\tau) = -(\beta/\delta)\tau^2 + O(\tau^2).$$

Note that t obtains non-degenerate maximum or minimum at $\tau = 0$.

The Jacobian along this solution is given by

$$\begin{pmatrix} 1 & \alpha \\ 2\beta\tau & 1 \end{pmatrix} + \cdots ,$$

so the multipliers are $1 \pm (2\alpha\beta\tau)^{1/2} + \cdots$, which means that the orbit is hyperbolic on one side and elliptic on the other side of $\tau = 0$.

Type II. Creation and Annihilation

Target. Periodic orbit such that $\lambda_\gamma = 1$.

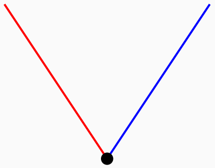


Figure 16: Creation and annihilation (Also called birth-death)

- The vector field develops an unstable periodic orbit •. It's a closing of a recurrent orbit, called **Pugh catastrophe**.
- • splits into a hyperbolic orbit and an elliptic orbit.
- In the phase portrait, we observe an elliptic orbit suddenly appeared. (**Creation**)
- If we read this in reversed time, we can observe a vanishing pair of orbits. (**Annihilation**)

III. Subtle doubling / halving
IV. Materialization / Murder

Periodic orbit such that $\lambda_\gamma = -1$

Meyer's Theorem - Transitional Fixed Point

We call (x, s) is a **transitional fixed point** if $\lambda_x = -1$ and satisfies some non-degeneracy condition. (See Appendix 2)

Theorem ([Mey70])

Let (x, s) be a transitional fixed point and x_t be an orbit cylinder near x_s, parametrized by $V \subset I$.

Then we can choose V small enough so that $V \setminus \{s\} = V_H \cup V_E$, where x_s is hyperbolic (elliptic) if $s \in V_H$ (V_E). Moreover, either one of the following holds.

1. *There exists functions $x_1^E, x_2^E : V_H \to U$ such that $(x_i^H(t), t)$ is elliptic periodic points of period 2 and $x_i^H(t) \to x$ as $t \to s$,*
2. *There exists functions $x_1^H, x_2^H : V_E \to U$ such that $(x_i^E(t), t)$ is hyperbolic periodic points of period 2 and $x_i(t) \to x$ as $t \to s$.*

We call these **an arc of nearby hyperbolic/elliptic periodic points**.

Type III. Subtle Doubling and Halving

Target. Periodic orbit such that $\lambda_\gamma = -1$.

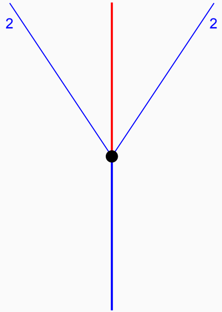


Figure 17: Subtle doubling and halving (Also called period doubling)

- The elliptic orbit cylinder becomes hyperbolic at •.
- An elliptic orbit with period 2 emerges.
- In phase portrait, we observe that the period of — is doubled. (**Subtle doubling**)
- In the opposite direction, the period is halved. (**Subtle halving**)

Type III. Subtle Doubling and Halving

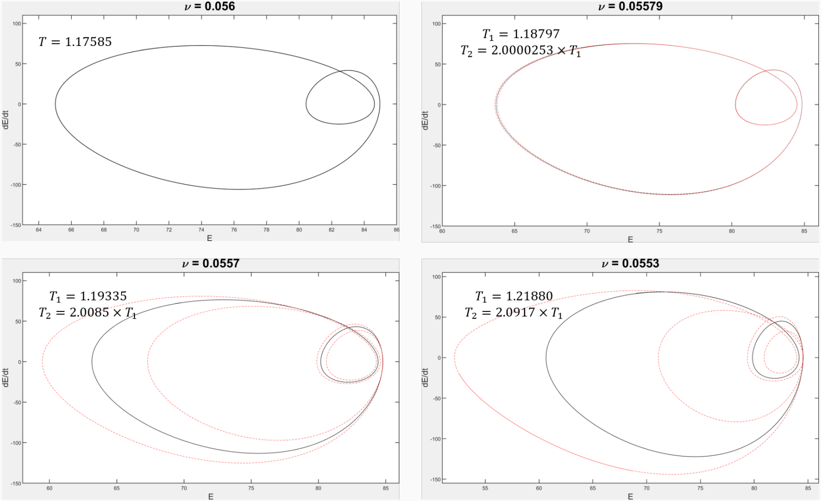


Figure 18: [Max] Period doubling in Kuramoto–Sivashinsky equation

Type IV. Materialization and Murder

Target. Periodic orbit such that $\lambda_\gamma = -1$.

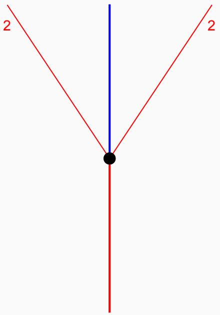


Figure 19: Materialization and murder

- Similar to subtle doubling, but with an interchange of elliptic and hyperbolic.
- We observe that — suddenly appeared. (**Materialization**)
- In the opposite direction, — suddenly disappears. (**Murder**)

V. Phantom Kiss
VI. Emission / Absorption

Periodic orbit such that $\lambda_\gamma^k = 1$, $k \geq 3$

Meyer's Theorem - Bifurcation Points

Let $\lambda_x = \exp(\pm 2\pi i l/k)$, $(l, k) = 1$, $k \geq 3$ and $0 < l < k$.

We call (x, s) is a k-**bifurcation point** if it satisfies some non-degeneracy condition. (See Appendix 3.)

Theorem ([Mey70])

Let (x, s) be k-bifurcation point. Then for a small orbit cylinder parametrized by $V \subset I$,

1. $(k = 3)$ *There exists three arcs of nearby hyperbolic periodic points of period 3, parametrized by $V \setminus \{s\}$, converging to x.*
2. $(k \geq 5)$ *There exists k arcs of nearby hyperbolic periodic points and k arcs of nearby elliptic periodic points of period k, parametrized by $V_+ \subset V \setminus \{s\}$, converging to x.*
3. $(k = 4)$ *One of $(k = 3)$ or $(k \geq 5)$-phenomena occurs.*

Genericity

Let $\mathcal{G}$ be the set of 1-parameter family of symplectomorphisms on S such that

1. Each periodic point is elliptic, hyperbolic, extremal or transitional.
2. If $\lambda_x = \exp(\pm 2\pi i l/k)$ for $k \geq 3$, then (x, s) is k-bifurcation point.

Theorem ([Mey70])

$\mathcal{G}$ is a residual set in the space of 1-parameter family of symplectomorphisms.

Type V. Phantom Kiss

Target. $\lambda_\gamma = \exp(2\pi i l/k)$, $(l,k)=1$ and $k=3$ or 4.

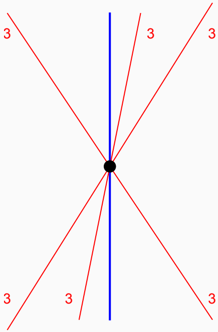


Figure 20: Phantom kiss
(Also called touch-and-go)

- A hyperbolic orbit of period 3 emerges in both directions of •.
- We only observe *a moment of instability* of —.
- The family of hyperbolic orbit only experiences a sudden absorption to the elliptic orbit. (**Phantom kiss**)

Type VI. Emission / Absorption

Target. $\lambda_\gamma = \exp(2\pi i l/k)$, $(l,k)=1$ and $k \geq 4$.

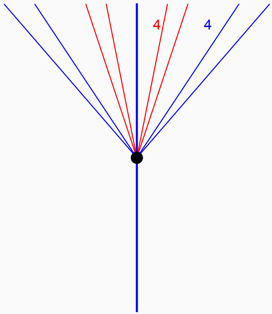


Figure 21: Emission and absorption

- A hyperbolic and an elliptic orbit of period k both emerges at $\bullet$. (**Emission**)
- An elliptic orbit of period k is observed.
- In the opposite direction, k-periodic orbits are absorbed into the original periodic orbit. (**Absorption**)

VII. Bubble / Liberation
VIII. Resonance

Critical point which is not $\mathcal{H}$-elementary

Change of Exponents of a Fixed Point

Target. Fixed point 0 of parametrized Hamiltonian H_μ.

Assume that 0 is elliptic if $\mu < 0$ and hyperbolic if $\mu > 0$, i.e.

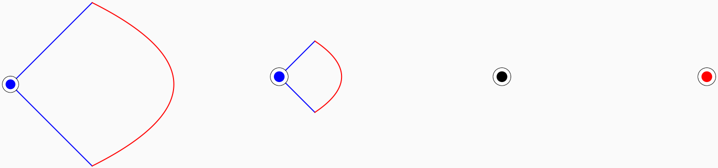
- $\mu < 0 : CE(x) = \{\pm i\alpha, \pm i\beta\}$ for $\alpha, \beta \in \mathbb{R}_{>0}$, $\alpha \neq \beta$.
- $\mu = 0 : \alpha = \beta$
- $\mu > 0 : CE(x) = \{\pm z, \pm \bar{z}\}$ where $z \notin S^1$.

If $\mu < 0$, the stable burst occurs, while nothing happens if $\mu > 0$.

Question. What happens to two elliptic orbit cylinders?

[MS71] showed that there are only two possibilities in general.

Type VII. Bubble

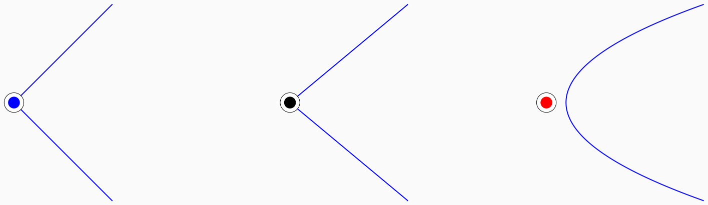


(a) $\mu \ll 0$ (b) $\mu < 0$ (c) $\mu = 0$ (d) $\mu > 0$

Figure 22: Bubble

The first possibility is a **bubble**: the shrink of two orbit cylinders into a fixed point.

Type VII. Liberation



(a) $\mu < 0$ **(b)** $\mu = 0$ **(c)** $\mu > 0$

Figure 23: Liberation

The second possibility is a **liberation**: two orbit cylinders patch together.

This phenomenom actually happens in CRTBP at L_4.

Type VIII. Resonance

Target. Fixed point 0 of parametrized Hamiltonian H_μ.

For characteristic exponents α_1 and α_2, if there exist coprime integers k, l such that

$$k\alpha_1 = l\alpha_2,$$

then $\mathcal{H}$-elementary assumption of Lyapunov theorem is violated.

This is called **resonance**, and there are many possible different phenomenom occurs depending on (k, l).

[SS73] gives the classification of generic resonance.

The situation is quite similar to the case of the bubble: 'bridges' between two orbit cylinders are collapsed into a point.

A Life of Orbit

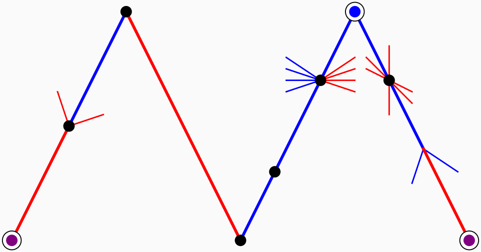


Figure 24: A life of an orbit

A periodic orbit can be born and die at critical point or other orbit.

For varying energy, it experiences various bifurcations, make birth of another orbits, and become from hyperbolic to elliptic or vice versa.

List of Generic Bifurcations

No.	Name	Case	Tool
I	Burst / Reincarnation	$\mathcal{H}$-elementary critical points	Lyapunov theorem
II	Creation / Annihilation	γ with $\lambda_\gamma = 1$	Extremal f.p.
III	Subtle Doubling / Halving	$\lambda_\gamma = -1$	Transitional f.p.
IV	Materialization / Murder	$\lambda_\gamma = -1$	Transitional f.p.
V	Phantom Kiss	$\lambda_\gamma^3 = 1$ $\lambda_\gamma^4 = 1$	3-bifurcation point 4-bifurcation point
VI	Emission / Absorption	$\lambda_\gamma^k = 1$ $(k \geq 4)$	k-bifurcation point
VII	Bubble / Liberation	$\mathcal{H}$-elementary violated	-
VIII	Resonance	$\mathcal{H}$-elementary violated	-

Closing

Relation with Floer Homology

Let γ_c be a periodic Hamiltonian orbit at the energy c.

The Conley-Zehnder index $\mu_{CZ}(\gamma_c)$ changes only if λ_c passes 1.

Indeed, $\mu_{CZ}(\gamma_c^N)$ changes if λ^N passes 1.

$\Rightarrow$ The bifurcation comes with the index change.

Also, the invariance of local Floer homology controls the index change.

Example : Hill's Lunar Problem

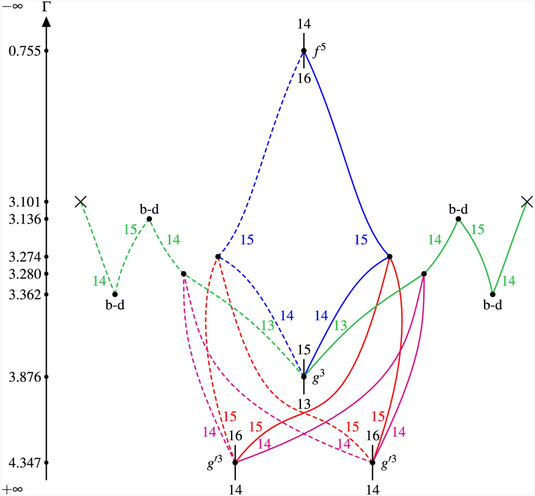


Figure 25: [Ayd23a] Bifurcation diagram of Hill's lunar problem

Higer Dimension?

Higher dimensional W

- [Mey70] said, there would be no essentially new generic behavior.
- The bifurcation would happen when one multiplier becomes a root of unity, and *generically* we don't expect the other multiplier simultaneously becomes another root of unity.

Higher dimensional I (the parameter of bifurcation)

- [Mey70] said, there are too many essentially different forms of bifurcation if we increase the dimension of parameters.
- The theory would become very wild, and there's no systemetic result as I know.

Genericity and Symmetry

Generically, a given Hamiltonian system is generic. (It's trivial.)

To check the genericity, which is well-defined in this case, we must check the linearized return map for a periodic orbit.

To say *'this system is generic'* is impossible in practice, because there are infinitely many periodic orbits in general.

Moreover, most of interesting systems are *not* generic, because of their symmetries in nature.

- The circular restricted three-body problem has $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmtery.
- [Ayd23b] The linear symmetries of Hill's lunar problem, a limit case of of CRTBP, is given by $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Many works were done for symmetric Hamiltonians, including [GSM87].

Pitchfork Bifurcation

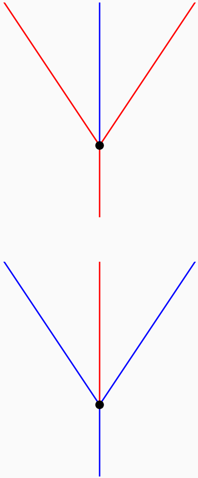


Figure 26: Pitchfork bifurcations

- The orbit changes hyperbolicity and ellipticity, and **two** orbits with **period 1** are born.
- The reason is $\mathbb{Z}_2$-symmtery of *the orbit* or the *return map*.
- This type of bifurcation, which is not *generic*, is observed in Hill's lunar problem, CRTBP, and many other systems with symmetry.

Further Questions

Investigating bifurcation by Conley-Zehnder index:

- Floer theory provides systemetic way to study bifurcation.
- [AFvK$^+$24] studied Hill's lunar problem with CZ index.
- We might apply the method to three-body problem and other interesting systems to investigate periodic orbits and bifurcations.

General relation between the change of index and the bifurcation type:

- [FKM23] showed the invariance of χ_{SFT} explicitly in each case.
- If we know the bifurcation type, what can we say about the index of specific orbits?

The End

Figure 27: Thank you for your attention!

Appendix 1. Twist Condition

$\Psi : 0 \in U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is α-**twist map** if 0 is a fixed point of Ψ with characteristic exponents $\pm i\alpha$ for $\alpha \in \mathbb{R}$.

If we can write

$$\Psi(z) = e^{i\alpha + \beta|z|} z + R_4(z),$$

where $R_4(z) \leq K|z|^4$ for some $K > 0$, we call such Ψ is written in (α, β)-**normal form**.

Theorem (Birkhoff-Sternberg-Moser Normal Form)

Let Ψ be α-twist mapping with $\alpha \neq 0$, $k\pi/2$ or $2k\pi/3$. Then there exists a chart such that Ψ can be written in (α, β)-normal form and the sign of β is independent of the change of the chart.

An α-twist map is an **elementary twist map** if $\alpha \neq 0$, $k\pi/2$, or $2k\pi/3$ and $\beta \neq 0$, which was mentioned in the Birkhoff-Lewis theorem.

Appendix 2. Transitional Fixed Point

A fixed point (x, s) is **transitional** if locally

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + (\text{h.o.t}),$$

and for

$$P_{qq} = 2\alpha, \quad P_{qp} = \beta, \quad P_{qt} = \gamma, \quad P_{qqq} = 6\delta, \quad Q_{qq} = 2\zeta$$

with $\eta = \delta + \alpha\zeta + \alpha\beta/2$,

$$\gamma \neq 0, \quad \alpha^2 + 2\eta \neq 0.$$

Appendix 3. Bifurcation points

Let $\lambda_x = \exp(\pm 2\pi i l/k)$, $k \geq 3$, $(l,k) = 1$ and $0 < l < k$.

[Bir27] introduced a normalization procedure, which gives local expression

$$
\begin{aligned}
(q,p) &= (r^{1/2}\cos\theta, r^{1/2}\sin\theta), \\
(Q,P) &= (R^{1/2}\cos\Theta, R^{1/2}\sin\Theta), \\
\Theta &= \theta + \frac{2\pi l}{k} + \frac{t\alpha(t)}{k} + \sum_{j=1}^{\lfloor (k-2)/2 \rfloor} \frac{\beta_j(t)}{k} r^j \\
&\quad + \frac{\gamma(t)}{k} r^{(k-2)/2}\cos k\theta + \Theta_1(\theta,r,t), \\
R &= r + \frac{2\gamma(t)}{k} r^{k/2}\sin k\theta + R_1(\theta,r,t).
\end{aligned}
$$

Appendix 3. Bifurcation points

Moreover, we require for $r^{1/2} = \rho$,

$$\frac{\partial^j R_1}{\partial \rho^j}(\theta, 0, 0) = 0 \quad \text{for } j = 0, 1, \cdots, k+1$$
$$\frac{\partial^i \Theta_1}{\partial \rho^i}(\theta, 0, 0) = 0 \quad \text{for } i = 0, 1, \cdots, k-1.$$

We denote $\alpha(0) = \alpha$, $\beta_1(0) = \beta$, $\gamma(0) = \gamma$. If

- ($k = 3$) $\alpha, \gamma \neq 0$,
- ($k = 4$) $\alpha, \gamma, \beta \pm \gamma \neq 0$,
- ($k \geq 5$) $\alpha, \beta, \gamma \neq 0$,

we call (x, s) a k-**bifurcation point**.

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