Bifurcation of Periodic Orbits in Hamiltonian Systems

Dongho Lee

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SNU, QSMS

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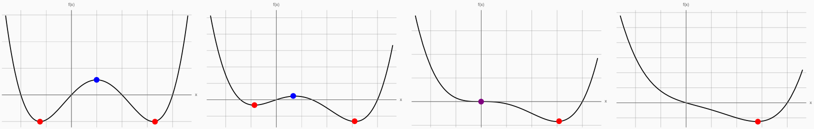
Introduction

Bifurcation : Toy Example

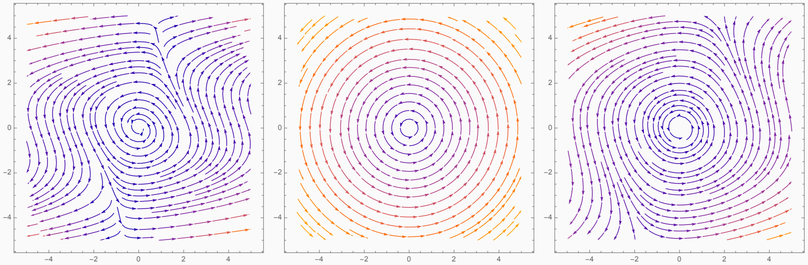


(a) $t = -1$ **(b)** $t = -0.5$ **(c)** $t = 0$ **(d)** $t = 1$

Figure 1: Graphs of $f_t(x) = x(x-2)(x^2+t)$

As t increases, there are changes of the critical points.

Bifurcation : Real Example



(a) $\mu < 0$ **(b)** $\mu = 0$ **(c)** $\mu > 0$

Figure 2: Trajectory of van der Pol oscillator, $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$.

As μ increases, there are changes of the periodic orbits.

Hamiltonian dynamics and symplectic manifolds

Classical Mechanics

Classical mechanics is governed by 2nd order ODE on $\mathbb{R}^n$ which is given by Newton's second law,

$$F = ma \quad \Leftrightarrow \quad -\nabla V(q) = \ddot{q},$$

where $q \in \mathbb{R}^n$ is the position and V is the **potential energy**.

The **mechanical energy** is defined by

$$H = K + V := \frac{|\dot{q}|^2}{2} + V(q),$$

where K is the **kinetic energy**.

Law (Classical Energy Conservation)

H is conserved along the trajectory of q determined by force F of the form $-\nabla V$ where $V = V(q)$.

Hamiltonian Dynamics

Idea. Using the energy conservation as a principal law.

Let $\omega = dp \wedge dq$, defined on $T^*\mathbb{R}^n$.

For $H : T^*\mathbb{R}^n \to \mathbb{R}$, we define a **Hamiltonian vector field** X_H by

$$i_{X_H}\omega = -dH.$$

Then,

- The flow of X_H preserves H, i.e. $X_H(H) = 0$.
- $\dot{x} = X_H(x)$ defines the mechanical system with total energy H. For example, if $H(q,p) = ||p||^2/2 + V(q)$, we have

$$\dot{q} = p, \quad \dot{p} = -\nabla V.$$

Symplectic Manifolds

We can generalize the concept to certain manifolds.

Let ω be a non-degenerate closed 2-form on a manifold W.

Then,

- $\ker \omega = 0 \quad \Rightarrow$ we can define X_H by $i_{X_H}\omega = -dH$.
- $\omega(X, X) = 0 \Rightarrow$ energy conservation, $\omega(X_H, X_H) = dH(X_H) = 0$.
- $d\omega = 0 \quad \Rightarrow$ invariance of the physical law, $\mathcal{L}_{X_H}\omega = 0$.

We call (W, ω) a **symplectic manifold**, and (W, ω, H) a **Hamiltonian system**.

Periodic orbits

Periodic orbit is an orbit γ of $Fl_t^{X_H}$ such that $\gamma(0) = \gamma(\tau)$ for some τ.

Why is this important?

- Invariant sets are building blocks of a dynamical system.
- **Arnold conjecture** : The minimal number of Hamiltonian periodic orbits are bounded by the sum of Betti numbers of the manifold.
- **Floer theory** : Periodic orbits are generators of $HF_*(M)$, an important symplectic invariant.
- Practical purposes.

Conley-Zehnder index

Poincaré Section and Return map

Theorem (Poincaré Section)

Let γ be a periodic orbit of X in W and $x \in \gamma$.

1. *There exists a codimension 1 submanifold $Y \subset W$ such that $x \in Y$ and is transversal to X.*
2. *In the case of Hamiltonian flow, we can take Y such that each $S_c = Y \cap H^{-1}(c)$ is a symplectic submanifold.*

We call Y or S a **Poincaré section***.*

We can define the **return map** by $\Psi(y) = Fl^X_{\tau(y)}(y)$ for y close to x where $\tau(y) = \min\{t > 0 \,:\, y \in Y\}$.

In the Hamiltonian setting, Ψ is a symplectomorphism.

Poincaré Section

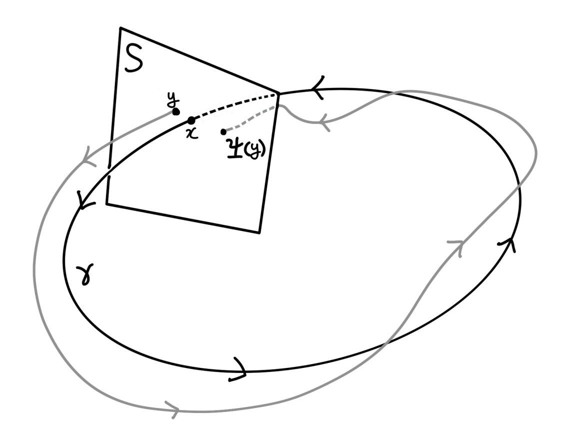


Figure 3: A Poincaré section and the return map

Elliptic and hyperbolic orbits

Let $\gamma(0) = \gamma(\tau) = x$ and Ψ be a return map.

Then Ψ is (locally) a **symplectomorphism**, which means $\omega = \Psi^*\omega$.

In local coordinates, we can write this condition as

$$A^T \Omega A = \Omega \quad \text{where } A = d\Psi_x,\ \Omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Note

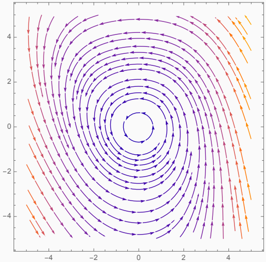
If $\lambda \in \sigma(A) = \{\text{eigenvalues of } A\}$, then $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1} \in \sigma(A)$.

If $|\lambda| \neq 1$ for any $\lambda \in \sigma(\gamma) = \sigma(d\Psi_x)$, we call γ **hyperbolic**.

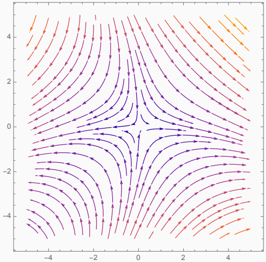
If $|\lambda| = 1$ for any $\lambda \in \sigma(\gamma)$, we call γ **elliptic**.

Elliptic and hyperbolic fixed points

Periodic orbits correspond to the fixed points of the return map.



(a) Elliptic fixed point



(b) Hyperbolic fixed point

Figure 4: Trajectory near the elliptic and hyperbolic fixed points.

Elliptic orbits are stable, while hyperbolic ones are not.

Elliptic and hyperbolic orbits in dimension 4

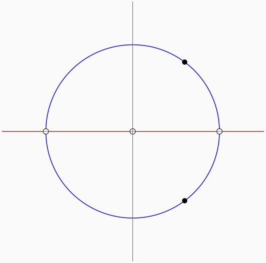
If $\dim M = 4$, the situation is simpler since there are only 2 eigenvalues.

- Elliptic: $\sigma(\gamma) = \{e^{i\alpha}, e^{-i\alpha}\}$ for $0 < \alpha < \pi$.
- Positive hyperbolic : $\sigma(\gamma) = \{r, r^{-1}\}$ for $r > 0$.
- Negative hyperbolic : $\sigma(\gamma) = \{-r, -r^{-1}\}$ for $r > 0$.
- Degenerate case: $\sigma(\gamma) = \{1, 1\}$ or $\{-1, -1\}$

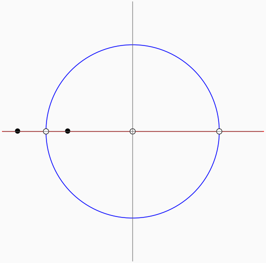
Note that if $\lambda \in \sigma(\gamma)$, then $\lambda^N \in \sigma(\gamma^N)$. Hence,

Simple orbit	**Odd cover**	**Even cover**
Elliptic	Elliptic	Elliptic
Positive hyperbolic	Positive hyperbolic	Positive hyperbolic
Negative hyperbolic	Negative hyperbolic	Positive hyperbolic

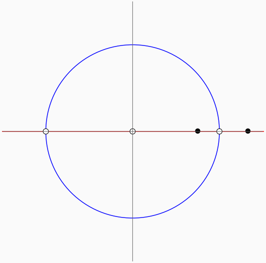
Elliptic and hyperbolic orbits in dimension 4



(a) Elliptic



(b) Negative hyperbolic



(c) Positive hyperbolic

Figure 5: Characteristic multipliers.

A transition between elliptic and hyperbolic happens through $\lambda = \pm 1$.

Remark

In generic dynamical system, every periodic orbit is hyperbolic. It means that there is an essential feature of Hamiltonian system distinguished from the generic dynamical systems.

Conley-Zehnder index

Let L_t be a linearized flow $dFl_t^{X_H}$ along γ.

After trivialization, this is a path in $Sp(2n)$.

At the points t such that $\det(L_t - \mathrm{Id}) = 0$, we define a **crossing form** by

$$Q_t(v) = \omega|_{\ker(L_t - \mathrm{Id})}(v, \dot{L}_t v).$$

The **Conley-Zehnder index** is

$$\mu_{CZ}(\gamma) = \frac{1}{2}\mathrm{Sign} Q_0 + \sum_{\det(L_t - \mathrm{Id})=0} \mathrm{Sign} Q_t + \frac{1}{2}\mathrm{Sign} Q_\tau.$$

Note that $\mu_{CZ}(\gamma)$ changes only if $1 \in \sigma(L_t)$.

Floer homology

The **Floer homology** is defined by a chain complex (C_*^F, ∂^F) where

- C_*^F is generated by Hamiltonian 1-periodic orbits.
- The generators are graded by $\mu_{CZ}(\gamma)$.
- ∂^F is defined by counting the **Floer cylinders**.

We can define a Floer homology for a certain energy hypersurface $H^{-1}(c)$, say SH_c, whose generators are *every* periodic orbit and their multiple covers with energy c.

The Floer homology is a symplectic invariant, and SH_c is invariant unless c passes the critical energy level.

Bifurcations of Hamiltonian orbits

Bifurcation

Two vector fields X, Y on W are **topologically conjugate** if there exists a homeomorphism $F : W \to W$ which carries the oriented orbits of X to the ones of Y.

It means that the dynamics of X and Y are essentially the same.

Consider a 1-parameter family of vector fields X_t on W. A point s is

- a **robust point** if X_s, X_t are topologically conjugate if $|s - t| < \varepsilon$,
- a **bifurcation point** if not.

Orbit cylinder

Setting. Hamiltonian vector field X_H on 4-dimensional (W, ω).

Consider a periodic orbit γ of period τ at energy level c.

We denote λ_c for the eigenvalue of L_τ of $\gamma_c \subset H^{-1}(c)$.

Theorem (Orbit cylinder)

If $\lambda_c \neq 1$, there exists an embedded cylinder $\Gamma : S^1 \times (c - \varepsilon, c + \varepsilon) \to W$ such that

1. $\Gamma(-, c) = \gamma(-)$.
2. $\Gamma(-, c') = \gamma_{c'}(-)$ *is a periodic orbit with energy* c'.
3. *Under an appropriate choice,* $\lambda_{c'}$ *changes smoothly along* c'.

Orbit cylinder

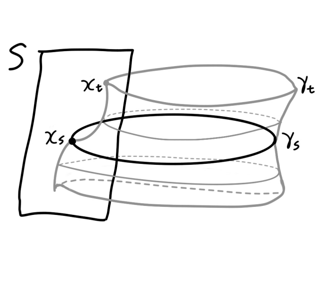


Figure 6: Orbit cylinder

Generically, if $\lambda_c \neq 1$, we can take a neighborhood of γ in W so that the only periodic orbit contained in the neighborhood with period close to τ are the ones in the orbit cylinder.

Bifurcation of Periodic Orbits

Consider the situation we increase the energy c.

- λ_c changes smoothly along c.
- If $\lambda_{c_0} = 1$, there can be another family of periodic orbits, say γ'_c, which is arbitrarily close to γ_{c_0}.
- On the other hand, the family γ_c might disappear at c_0.

These phenomena are the **bifurcation of periodic orbits**.

- If γ'_c exists for $c < c_0$, we say γ'_c **disappears** at c_0.
- If γ'_c exists for $c > c_0$, we say γ'_c **emerges** at c_0.

Bifurcation : Some Observations

Let γ_c be a family of simple periodic orbit.

- If $\lambda_{c_0} = 1$, the bifurcation may occur. In this case, even the core family γ_c may disappear.
- If $\lambda_{c_0}^N = 1$, the bifurcation may occur at $\gamma_{c_0}^N$. In this case, γ_c^N cannot disappear since γ_c still exists.
- The change between elliptic and hyperbolic occurs only if $\lambda_c = \pm 1$.

$\Rightarrow$ We need to see the cases $\lambda^N = 1$.

Classification : Observations

Theorem ([Mey70])

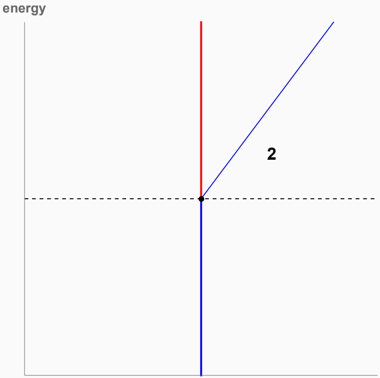
Generically, there are 8 types of bifurcations of Hamiltonian system. In particular, 5 of them are related with the periodic orbits.

- $\lambda = 1$: *Birth-death*
- $\lambda = -1$: *Period doubling, murder*
- $\lambda^3 = 1$ *or* $\lambda^4 = 1$: *Phantom kiss*
- $\lambda^N = 1$, $N \geq 4$: *Emission*

Here, we assume that $\lambda = \exp(2\pi i k/N)$ *and* $(k, N) = 1$.

Classification : Diagram

We'll see these kinds of diagrams.



- Vertical axis is the energy.
- • is the bifurcation point.
- Each line is a family of periodic orbits.
- — are elliptic, — are hyperbolic.
- The number (**2**) is the approximate period of the orbit compare to the orbit at the bifurcation point.

Birth-death

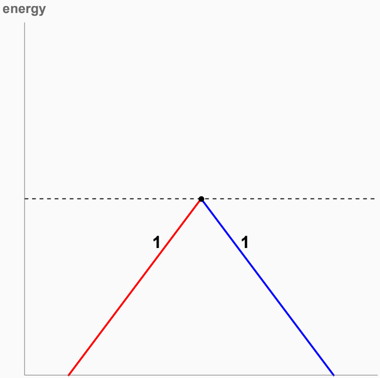


Figure 7: Birth-death ($\lambda = 1$)

- At c_0, a hyperbolic orbit and an elliptic orbit meets and disappears.
- After c_0, a non-periodic recurrent orbit appears, called **Pugh catastrophe**.
- [AM78] called this *creation and annihilation*.

Period Doubling

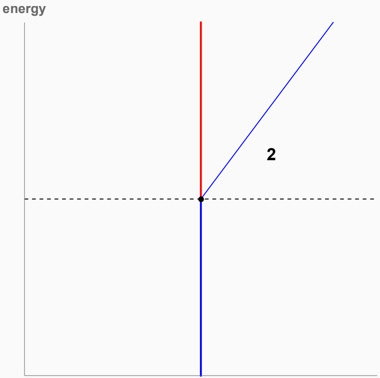


Figure 8: Period doubling ($\lambda = -1$)

- At c_0, an elliptic orbit becomes negative hyperbolic.
- Its double cover changes from elliptic to positive hyperbolic.
- A family of elliptic orbits with period 2 appears.
- The inverse is called **period halving**.
- [AM78] called this *subtle doubling* and *halving*.

Period Doubling

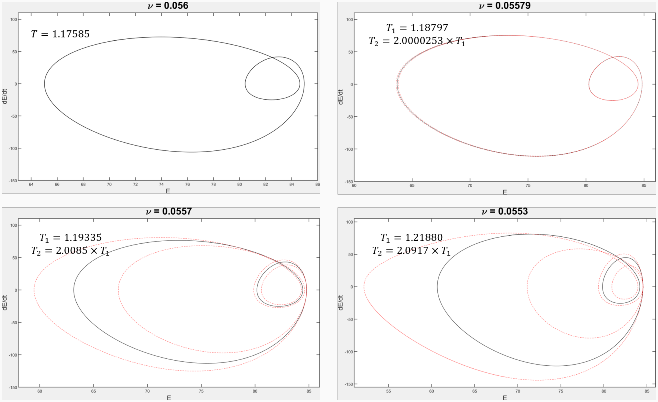


Figure 9: Example of a period doubling

Murder

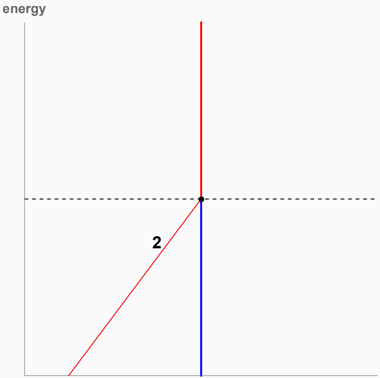


Figure 10: Murder ($\lambda = -1$)

- At c_0, a negative hyperbolic orbit becomes elliptic.
- Its double cover changes from positive hyperbolic to elliptic.
- A family of hyperbolic orbits with period 2 disappears.
- The inverse is called **materialization**.

Phantom kiss

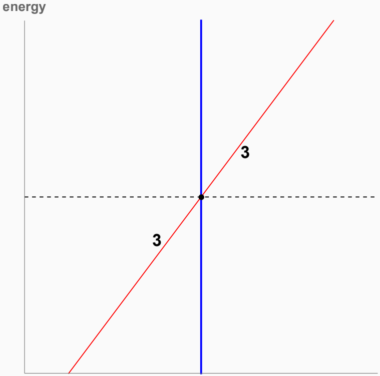


Figure 11: Phantom kiss ($\lambda^3 = 1$ or $\lambda^4 = 1$)

- This only happens for $N = 3, 4$.
- At c_0, a family of hyperbolic orbits of period N disappears, but emerge right after c_0.
- The core orbit and its N-th cover are elliptic.
- It might be understood as two families meets and become degenerate at one point.
- This is also called *phantom kiss*.

Emission

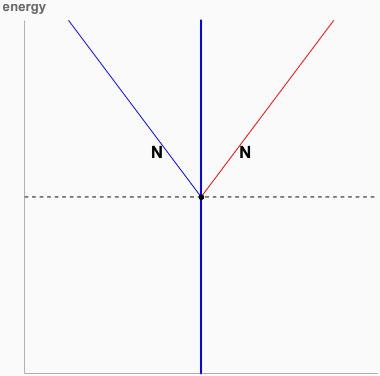


Figure 12: Emission ($\lambda^N = 1$, $N \geq 4$)

- At c_0, a family of elliptic orbits and a family of hyperbolic orbits, both have period N, emerge.
- The core orbit and its N-th cover are elliptic.
- The inverse is called *absorbtion*.
- At $N = 4$, both phantom kiss and emission can happen, depend on the ratio of specific terms of the return map.

Relation with Conley-Zehnder index

As c increases, $\mu_{CZ}(\gamma_c)$ changes only if $\lambda_c = 1$.

If $\lambda_c^N = 1$, $\mu_{CZ}(\gamma_c^N)$ might change.

$\Rightarrow$ Change of $\mu_{CZ}(\gamma)$ and bifurcation occur simultaneously!

Example. Consider the phantom kiss of γ^3 at c_0 with hyperbolic orbit γ'.

Passing c_0, $\mu_{CZ}(\gamma^3)$ changes by ± 2. (elliptic $\to$ elliptic)

Let's assume $\mu_{CZ}(\gamma^3_{c_-}) = N$, $\mu_{CZ}(\gamma^3_{c_+}) = N + 2$.

From the invariance of the local Floer homology, we can conclude $\mu_{CZ}(\gamma') = N + 1$ and they cancel each other.

Three-body problem

Kepler problem

The **Kepler problem** describes the motion of an object under the gravitational force of the other object, defined by a Hamiltonian

$$\begin{aligned} E : T^*(\mathbb{R}^3 \setminus \{0\}) &\to \mathbb{R} \\ (q, p) &\mapsto \frac{1}{2}|p|^2 - \frac{1}{|q|}. \end{aligned}$$

If $E < 0$, every orbit is an ellipse with a focus at the origin.

Three-body problem

The **restricted three-body problem** describes the motion of a mass-less body under the gravitational force of two bodies, say E of mass $1-\mu$ (earth) and M of mass μ (moon), and is defined by Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - M(t)|} - \frac{1-\mu}{|q - E(t)|}.$$

The motion of $E(t)$ and $M(t)$ are governed by the Kepler problem.

If we assume the Kepler orbit to be circular, we get **circular restricted three-body problem**

$$H(q,p) = \frac{1}{2}|p|^2 + (p_1 q_2 - q_1 p_2) - \frac{\mu}{|q - (1-\mu, 0, 0)|} - \frac{1-\mu}{|q - (-\mu, 0, 0)|}.$$

See [FvK18] or [Cel10] for better understanding.

Three-body problem

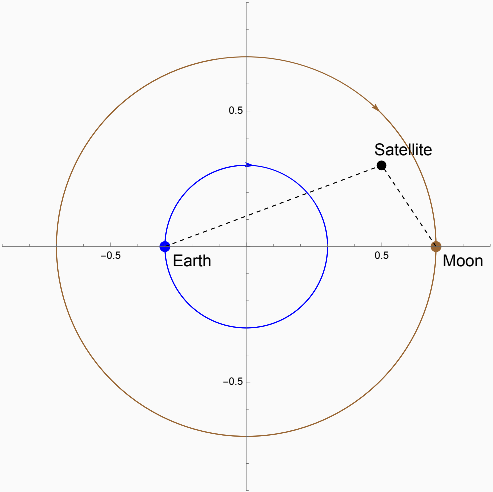


Figure 13: Illustration of the restricted three-body problem

Hill's region and Lagrange points

We can re-formulate the Hamiltonian as

$$H(q,p) = \frac{1}{2}|\tilde{p}|^2 - \left(\frac{1}{2}|q|^2 + \frac{\mu}{r_M} + \frac{1-\mu}{r_E}\right) := \frac{1}{2}|\tilde{p}|^2 + U(q).$$

We call $U(q)$ **effective potential**.

For an energy c, we have $H(q,p) = c$ iff $U(q) \le c$.

We call $\{q : U(q) \le c\}$ a **Hill's region**.

$U(q)$ has 5 critical points, and the topology of Hill's region changes through these critical energy.

The critical points are called **Lagrange points**.

Hill's region and Lagrange Points

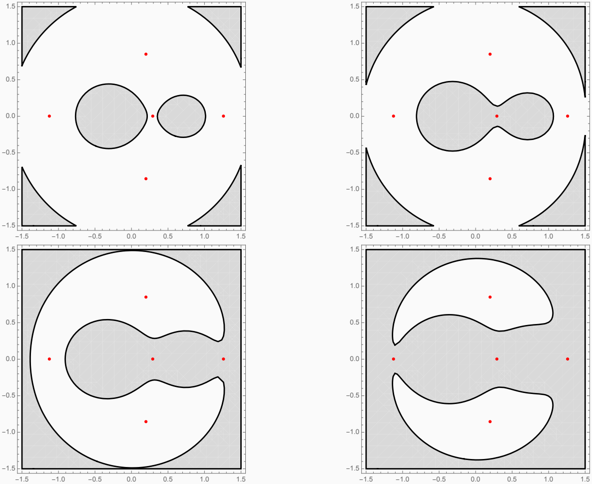


Figure 14: Hill's region for various energies

Limit case : Rotating Kepler problem

The **rotating Kepler problem** is the case $\mu = 0$, defined by

$$H(q,p) = \frac{1}{2}|p|^2 + (p_1 q_2 - p_2 q_1) - \frac{1}{|q|}.$$

Important orbits : these families exists for every energy $c < -3/2$.

- Planar circular orbits $\gamma_\pm$ (retrograde, direct)
- Vertical collision orbits $\gamma_{c_\pm}$ (northern, southern)

Limit case : Rotating Kepler problem

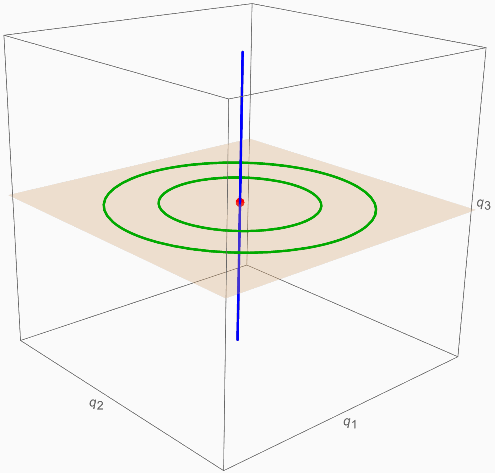


Figure 15: 4 periodic orbits of the rotating Kepler problem

Conley-Zehnder indices of rotating Kepler problem

Theorem ([Lee25], ArXiv Preprint)

Let $E_{k,l} = -0.5(k/l)^{2/3}$, $c^{\pm}_{k,l} = E_{k,l} \pm 1/\sqrt{-2E_{k,l}}$. *Then,*

$$\mu_{CZ}(\gamma_+^N) = \begin{cases} 4N-2 & \textit{if} \quad c < c^+_{N-1,1} \\ 4N-2-4k & \textit{if} \quad c^+_{N-k+1,k-1} < c < c^+_{N-k,k} \\ 2 & \textit{if} \quad c > c^+_{1,N-1} \end{cases},$$

$$\mu_{CZ}(\gamma_-^N) = \begin{cases} 4N+2 & \textit{if} \quad c < c^-_{N+1,1} \\ 4N+2+4k & \textit{if} \quad c^+_{N+k,k} < c < c^+_{N+k+1,k+1} \end{cases},$$

$$\mu_{CZ}(\gamma_{c_\pm}^N) = 4N.$$

Note. For low energy, these 4 orbits and their multiple covers gives every generator of $SH(T^*S^3)$ up to certain degree.

Bifurcation of the rotating Kepler problem

At $c^{\pm}_{k,l}$, an S^3-family of orbits emerge at γ_+^{k+l} and disappears at g_-^{k-l}.

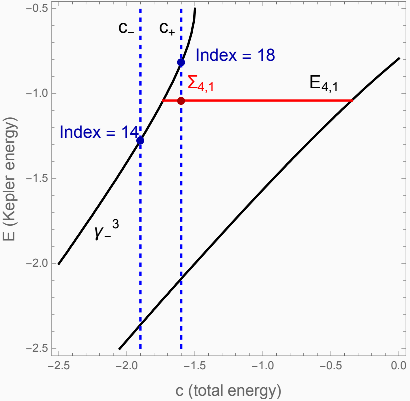


Figure 16: Bifurcation of the rotating Kepler problem

This non-genericity occurs because of the strong symmetry of the system.

Further works

- Computing of the Conley-Zehnder index and relate this with the bifurcation of the integrable systems.
- Analyzing bifurcation behavior of non-integrable systems, in particular the system related to the three-body problem.
- Reveal the relation between symmetry-breaking of the system and the change of bifurcation type.
- Find generic relation between the bifurcation and Conley-Zehnder index, and moreover the wall-crossing phenomena of the Floer homology.

Thank you for your attention!

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