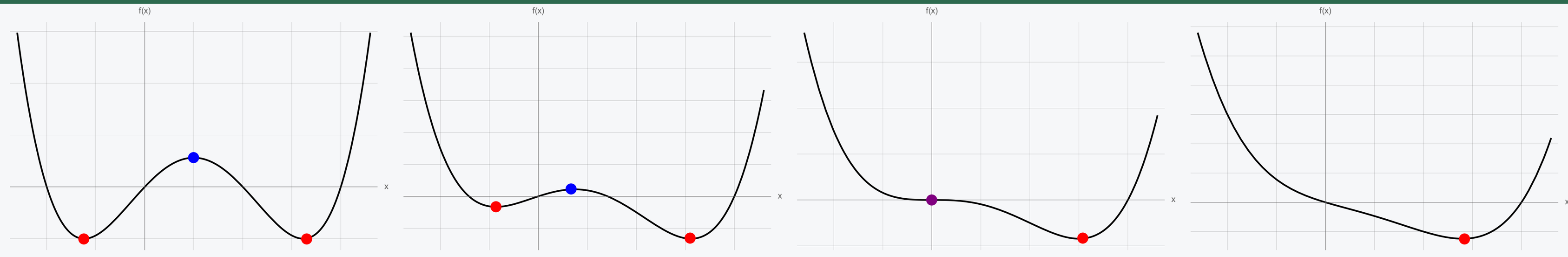
Periodic Orbits of Rotating Kepler Problem : Bifurcation and Conley-Zehnder Indices

Dongho Lee, QSMS, Seoul National University

Presented at **Celestial Mechanics & N-body Dynamics**, Tokyo, Japan, 15 November 2025

Visit my Homepage!

Toy Example : Function



This is graph of $f_t(x) = x(x^2+t)(x-1)$ for $t = -1, -1/2, 0$ and 1.
See the change of critical points.

- $t < 0$, two minima (●) and one maximum. (●)
- $t = 0$, one minimum and one maximum meets at the inflection point. (●)
- $t > 0$, one minimum survives. (●)

This phenomenom also happens for **periodic orbits**.

Bifurcation of Hamiltonian Periodic Orbits

Let $H : W \to \mathbb{R}$ be a Hamiltonian and $\gamma_c \subset H^{-1}(c)$ be a periodic orbit with energy c.
We call γ_c is **non-degenerate** if the linearized return map on the Poincaré section doesn't have 1 as an eigenvalue.
(Similar to critical point x of a function f such that $f''(x) \neq 0$.)
If γ_c is non-degenerate, there exists a family of periodic orbits near energy c,

$$\Gamma = \{(\gamma_{c'}, c') \,:\, \gamma_{c'} \subset H^{-1}(c'),\ c-\varepsilon < c' < c+\varepsilon\}.$$

Topologically, Γ is a cylinder, i.e., $\Gamma : S^1 \times (c-\varepsilon, c+\varepsilon) \to W$.
If γ_c is degenerate, **bifurcation** might happen: γ_c vanishes or another orbits are born.
Generically, there are 8 types of bifurcations of Hamiltonian periodic orbits. ([AM78])
Another kind of bifurcation may happen if there is enough symmetry. (e.g. RKP)

Conley-Zehnder Index and Floer Homology

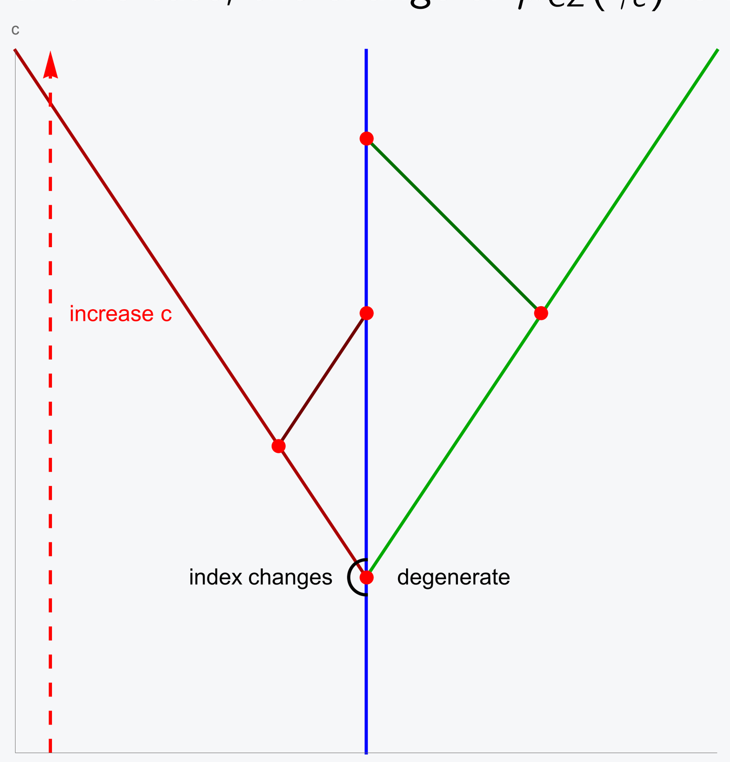
The **Conley-Zehnder index** $\mu_{CZ}(\gamma)$ is an integer associated to the periodic orbit. ([CZ83])
If the energy hyperplane $H^{-1}(c)$ has a contact structure, the **symplectic homology** $SH_*(W_c)$, generated by periodic orbits, can be defined for each energy c. ([FH94], [CFH95])
The circular restricted three-body problem have such property. ([AFvKP12], [CJK20])

- $SH_*(W_c)$ is invariant of c, until c passes a critical energy.
- The Conley-Zehnder index gives a grading to $SH_*(W_c)$.

It means that the indices of *all* periodic orbits are globally controlled.
Computing tool : Local Floer homology, Morse-Bott spectral sequence.

Index and Bifurcation

$\mu_{CZ}(\gamma_c)$ is invariant while γ_c is non-degenerate, and might change when γ_c degenerates.
In this case, the change of $\mu_{CZ}(\gamma_c)$ is related to the index of other orbits.



Toy model of a bifurcation

Periodic orbits with same energy c has the same y-coordinates.
There's only one orbit ● for low c.
When ● passes degeneracy (●), new orbits (●, ●) are born.
$SH(W)$ generated by ● at low energy and ●, ●, ● at higher energy must coincide, so the Conley-Zehnder indices of these orbits are related.
This happens until c passes critical energy.

Periodic Orbits of Kepler Problem

Kepler problem is defined by Hamiltonian $E : T^*\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$,

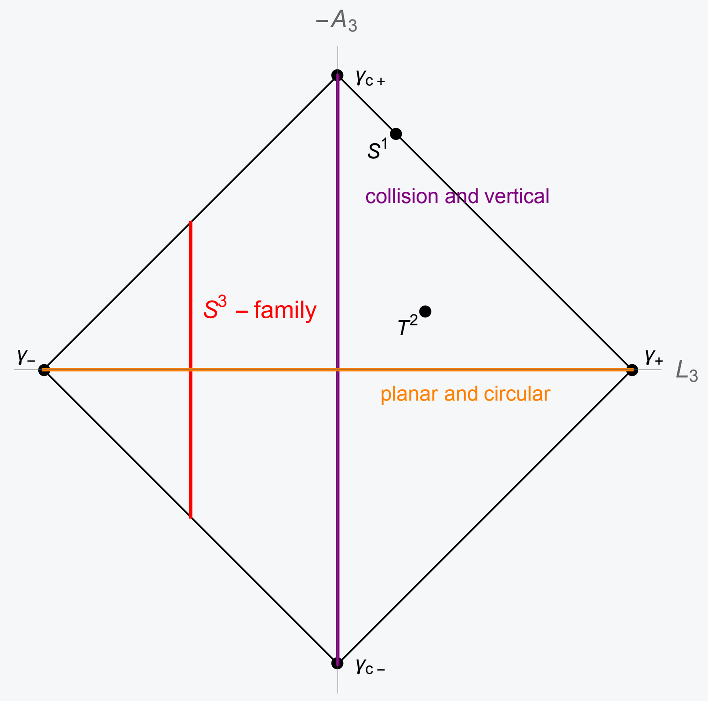
$$E(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|}.$$

If $E < 0$, every orbit is an ellipse with one focus at the origin.
We resolve the singularity at 0 by include collision orbits via **Moser regularization**.
The orbits have two invariants: angular momentum L and **Laplace-Runge-Lenz vector**

$$A = p \times L - \frac{q}{|q|}.$$

The space of periodic orbits with Kepler energy E can be parametrized by a map

$$\Phi : \mathcal{M}_E \to S^2 \times S^2$$
$$\gamma \mapsto (\sqrt{-2E}L - A, \sqrt{-2E} + A)$$



The diagram is an image of the map

$$\mathcal{M}_E = S^2 \times S^2 \to \mathbb{R}^2$$
$$\gamma = (x,y) \to \frac{1}{2}(x_3+y_3, x_3-y_3).$$

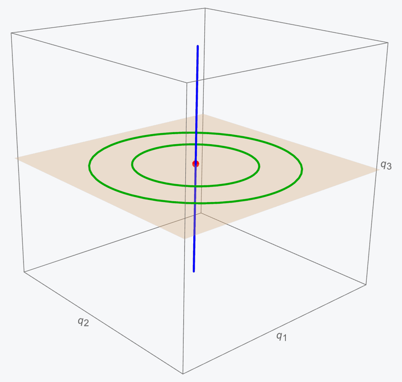
Four vertices correspond to the retrograde, direct and vertical collision orbits.
The family of orbits with same L_3 forms S^3-family.
The space is same as the space of unit geodesics of round S^3.

Periodic Orbits of Rotating Kepler Problem

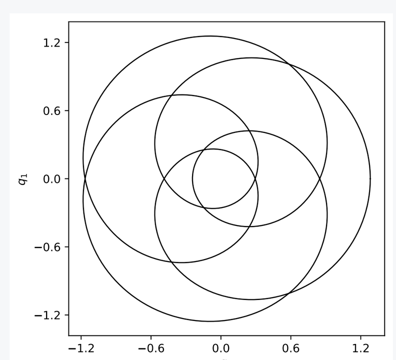
The **rotating Kepler problem** is defined by Hamiltonian

$$H(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|} + (p_1q_2 - p_2q_1) = E + L_3.$$

This is a limit $\mu = 0$ of the circular restricted three-body problem (CRTBP).



Since $\{E, L_3\} = 0$, the orbits which is invariant under the rotation along q_3-axis are always periodic.
Planar circular (○): **retrograde** (γ_+) and **direct** (γ_-).
Vertical collision (−): northern (γ_{c_+}) and southern (γ_{c_-}).
These are non-degenerate for generic energy $c < -1.5$.

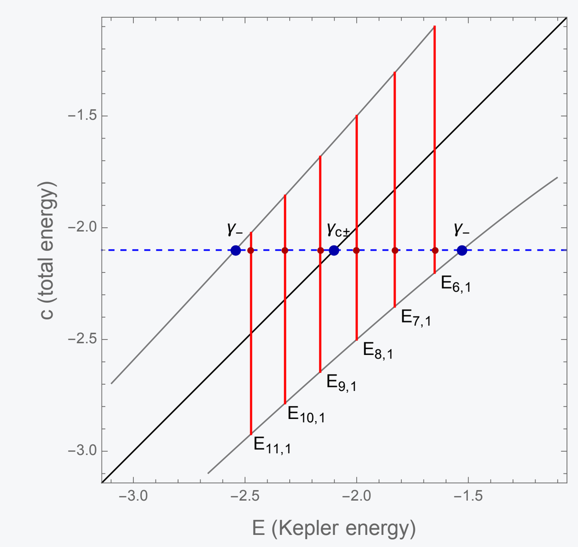


For $E_{k,l} = -0.5(k/l)^{2/3}$ for $k, l = 1, 2, \cdots$, the Kepler flow and rotation resonate and every orbit becomes periodic.
These orbits form S^3-family $\Sigma_{k,l}$ for each $E_{k,l}$.
At these energies, $\gamma_\pm$ and $\gamma_{c_\pm}$ become degenererate.
$\Sigma_{k,l}$ is born at γ_-^{k-l} and vanishes at γ_+^{k+l}.

- The simple retrograde orbit doesn't bifurcate if $c < -1.5$.
- N-th cover of retrograde orbit bifurcate $(N-1)$-times.
- Every cover of direct orbit bifurcate infinitely many times.
- Vertical collision orbits don't bifurcate, due to too much symmetry.

[AFFvK13] analyzed the planar case.
This is also observed in CRTBP, but there are much less symmetry.
$\Sigma_{k,l}$ become a finite set of discrete orbits, and the vertical collision also bifurcates.

Conley-Zehnder Indices of Rotating Kepler Problem



At generic energy c, the energy hyperplane $H^{-1}(c)$ $(\cdots)$ contains following orbits:

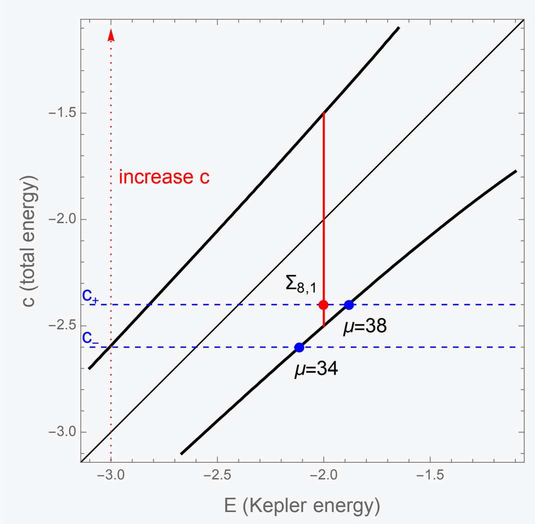
- 4 non-degenerate orbits: $\gamma_\pm$ and $\gamma_{c_\pm}$.
- $\Sigma_{k,l}$-family for each $E_{k,l}$ such that
$$E_{k,l} - (-2E_{k,l})^{-1/2} < c < E_{k,l} + (-2E_{k,l})^{-1/2}.$$
- These are all generators of $SH^{S^1,+}(T^*S^3)$, and their indices can be investigated with local Floer homology.
- $SH^{S^1,+}(T^*S^3)$ has one generator at degree 2 and two generators at each degree $2k \geq 4$.

Theorem ([Lee25], ArXiv Preprint)

Let $c_{k,l}^{\pm} = E_{k,l} \pm (-2E_{k,l})^{-1/2}$, and denote N-th cover of orbit γ by γ^N.

- $\mu_{CZ}(\gamma_+^N) = 4N - 2$ *for* $c < c_{N-1,1}^+$, *and decreases by* 4 *each time* c *passes* $c_{N-k,k}^+$.
- $\mu_{CZ}(\gamma_-^N) = 4N + 2$ *for* $c < c_{N+1,1}^-$, *and increases by* 4 *each time* c *passes* $c_{N+k,k}^-$.
- $\mu_{CZ}(\gamma_{c_\pm}^N) = 4N$ *for any* $c < -1.5$.
- $\mu_{RS}(\Sigma_{k,l}) = 4k - 1/2$.

ArXiv link



This behavior can be analyzed via local Floer homology and Morse-Bott spectral sequence.
c passes $c_{k,l}^-$: $\mu_{CZ}(\gamma_-^{k+l})$ increases by 4, $\Sigma_{k,l}$ is born.
c passes $c_{k,l}^+$: $\mu_{CZ}(\gamma_+^{k-l})$ decreases by 4, $\Sigma_{k,l}$ vanishes.
The change of $\mu_{CZ}(\gamma_\pm^N)$ must change $SH_*(W)$, but $\Sigma_{k,l}$ corrects the difference.

Discussion : Finding Periodic Orbits

[AFvK+] analyzed Hill's lunar problem in this framework with numerical computation.
Goal: Apply this framework to analyze the periodic orbits of circular restricted three-body problem, and other interesting systems.

References

[AFFvK13] Peter Albers, Joel W. Fish, Urs Frauenfelder, and Otto van Koert. The Conley-Zehnder indices of the rotating Kepler problem. *Math. Proc. Cambridge Philos. Soc.*, 154(2):243–260, 2013.

[AFvK+] Cengiz Aydin, Urs Frauenfelder, Otto van Koert, Dayoung Koh, and Agustin Moreno. Symplectic geometry and space mission design.

[AFvKP12] Peter Albers, Urs Frauenfelder, Otto van Koert, and Gabriel P. Paternain. Contact geometry of the restricted three-body problem. *Comm. Pure Appl. Math.*, 65(2):229–263, 2012.

[AM78] Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics.* Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, MA, second edition, 1978. With the assistance of Tudor Raţiu and Richard Cushman.

[CFH95] K. Cieliebak, A. Floer, and H. Hofer. Symplectic homology. II. A general construction. *Math. Z.*, 218(1):103–122, 1995.

[CJK20] WanKi Cho, Hyojin Jung, and GeonWoo Kim. The contact geometry of the spatial circular restricted 3-body problem. *Abh. Math. Semin. Univ. Hambg.*, 90(2):161–181, 2020.

[CZ83] C. C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold. *Invent. Math.*, 73(1):33–49, 1983.

[FH94] A. Floer and H. Hofer. Symplectic homology. I. Open sets in $\mathbf{C}^n$. *Math. Z.*, 215(1):37–88, 1994.

[Lee25] Dongho Lee. Conley-zehnder indices of spatial rotating kepler problem, 2025.

Download this poster