Dynamical Systems

① Smale "Differentiable dynamical systems" (1967)
② Abraham, Marsden "Foundations of Mechanics" (1978)
③ Celletti "Stability and Chaos in Celestial Mechanics" (2010)

(Smale) Lie group action $G \curvearrowright M$
$G = \mathbb{Z}$: discrete dynamics (1 diffeo. $F: M \to M$)
$G = \mathbb{R}$: continuous dynamics (1-param family $F_t : M \to M$)

$\boxed{\text{Topics}}$

① Invariant subsets. $F(S) \subset S$.
- Fixed point (rest point, equilibrium)
- Periodic point / Periodic orbits.
- And more ...

② Stability (of system)
- Give an eq. rel. $\sim_E$ on $\mathrm{Diff}(M)$
- If $\exists$ nbhd $\mathcal{U}$ of $F \in \mathrm{Diff}(M)$ s.t. $F' \in \mathcal{U} \Rightarrow F \sim_E F'$, we call F is E-stable. (Similar for vector fields)

Ex Top. conjugacy : $\exists h \in \mathrm{Homeo}(M)$ s.t. $F \circ h = h \circ F'$.

- "Generic" properties of diffeo / vfs. (Morse-Smale, etc.)
Hamiltonian dynamics : not generic, but still can be investigated.

③ Bifurcation

$\mu : I^n \longrightarrow \mathfrak{X}(M)$, $\mu(c) = X_c$

If $\exists$ nbhd $c \in U \subset C$ s.t. $c' \in U \Rightarrow X_c \sim_{top.} X_{c'}$, call c robust point

Otherwise, call c bifurcation point, denoted $B \subset I^n$.

- Through bifurcation, the fixed pts & per. orbits experiences a qualitative change.
- $\exists$ "Generic" behaviors.

④ More ...

- Hamilton–Jacobi formalism.
- Celestial mechanics.
- Non-Hamiltonian dynamics
- ...

Invariant Sets (See Ch. 6, [AM78])

Assume X is complete vector field, $\varphi_t = Fl_t^X$.

Orbit of $p \in M$ is $\quad \mathcal{O}(p) = \{\varphi_t(p) : t \in \mathbb{R}\}$.
$$\mathcal{O}^{\pm}(p) = \{\varphi_t(p) : \pm t \geq 0\}.$$

Fixed point $\varphi_t(p) = p \quad \forall t \in \mathbb{R}$.
Periodic point $\varphi_{t+\tau}(p) = p$ for some $\tau > 0$.
} Critical elements, denote Γ_X.
Closed orbit $\gamma = \mathcal{O}(p)$ where p is periodic pt.

Minimal set $\emptyset \neq S \subset M$ is minimal closed invariant subset.
- Critical elements are minimal.

ω-limit set $\quad \omega(p) = \bigcap_{n \in \mathbb{N}} \mathrm{cl}\,\{\varphi_t(p) : t \geq n\}$
α-limit set $\quad \alpha(p) = \bigcap_{n \in \mathbb{N}} \mathrm{cl}\,\{\varphi_t(p) : t \leq -n\}$
$\to$ If M is compact, closed, invariant, connected, nonempty.
$\Rightarrow$ Contains minimal set.

Denote $\quad \Lambda_X = \bigcup_{p \in M} \omega(p) \cup \alpha(p)$.

Prop $\quad \omega(p) = \{q \in M : \exists t_n \to \infty \text{ s.t. } \varphi_{t_n}(p) = q\}$
$\alpha(p) = \{ \quad '' \quad : \exists t_n \to -\infty \quad '' \quad \}$

Thm (Schwartz)
Let M be compact orientable 2-manifold, $A \subset M$ minimal.
Then $A \in \Gamma_X$ or $A = M \cong T^2$.
⌈Nontrivial part: If A is nowhere dense, $A \in \Gamma_X$.
∂A is closed invariant, so $\partial A = \emptyset$. $\boxed{\times}$|

$p \in M$ is <u>non-wandering</u> if $\forall t_0 > 0$ and nbhd U,
$\exists t > t_0$ s.t. $U \cap \varphi_t(U) \neq \emptyset$. Denote the set Ω_X.

<u>Prop</u> Ω_X is closed invariant and $cl(\Lambda_X) \subset \Omega_X$.

⌜Let $p_\infty \in \Lambda_X$, so $p_\infty = \lim \varphi_{t_n}(p) = \lim p_n$.
Take nbhd U of p_∞, so $\exists N$ s.t. $p_n \in U$ if $n \geq N$.
$\forall t_0 \geq 0$, $\exists t = t_{n_1} - t_{n_2} > t_0$ s.t. $p_{n_1} \in U \cap \varphi_t(U)$ since $p_{n_1} = \varphi_t(p_{n_2})$,
i.e. $p_\infty \in \Omega_X$. and $n_1 > n_2 \geq N$ $\boxtimes$⌟

<u>Hull</u> of p is $H(p) = cl(o(p)) = o(p) \cup \omega(p) \cup \alpha(p)$. $\Rightarrow$ invariant.
If $H(p)$ is compact, call p <u>compact</u>

$p \in M$ is <u>recurrent</u> if $\forall$ nbhd U, $h_U = \{t \in \mathbb{R} : \varphi_t(p) \in U\} \subset \mathbb{R}$,
$\exists \tau > 0$ s.t. $\forall \alpha \in \mathbb{R}$, $[\alpha, \alpha+\tau] \cap h_U \neq \emptyset$. Denote the set R_X.

p is <u>pseudorecurrent</u> if $p \in \omega(p) \cap \alpha(p)$.

<u>Thm</u> (Birkhoff)

(1) $H \subset M$ is cpt minimal $\Leftrightarrow$ $H = H(p)$ for some cpt recurrent p.
(2) $H \subset M$ is cpt minimal and $p \in H \Rightarrow p$ is cpt recurrent.
(3) $\emptyset \neq H \subset M$ is minimal $\Leftrightarrow H = H(p)$ $\forall p \in H$.
(4) $\Gamma_X \subset R_X \subset \Omega_X$.

⌜Let $p \in H$.
① H is invariant, so $o(p) \subset H \Rightarrow H(p) \subset H$ is compact.
Since $H(p)$ is invariant, by minimality, $H = H(p)$.
Let p be <u>not</u> rec., so $\exists U$ s.t. $\forall n$, $\exists \alpha_n$ s.t. $[\alpha_n, \alpha_n + n] \cap h_U = \emptyset$.
By compactness, $p_n = \varphi_{\alpha_n + n/2}(p)$ has conv. subseq. to p_∞.
Then $\varphi_t(p_n) \notin U$ for $-n/4 \leq t \leq n/4$, so whole orbit of p_∞
does not meet U. ↯ to minimality. $\boxtimes$⌟

Stability of orbits (AM Ch. 2.1, 6.3, C. Ch 12)

An orbit at p is $\pm$-orbitally stable if $d(p,p') < \delta \Rightarrow d(O^{\pm}(p), O^{\pm}(p')) < \varepsilon$
$(O^{\pm}(p))$
asymptotically stable if $\lim_{t\to\pm\infty} d(O^{\pm}(p), p'_t) = 0$.
Lyapunov stable if $\lim_{t\to\pm\infty} d(p_t, p'_t) = 0$.

If p is fixed pt, asymp. = Lyapunov = attractor

(Assume $M = \mathbb{R}^n$)
X defines system $\dot{p} = X(p) := F(p)$
Jacobian $J = (\partial_i F_j)$ gives linearized system $\dot{y} = J(p)y$ at each p.

Thm (Hartman-Grobman)
If p is a fixed point, the system is locally top. conj. to the linearized system.

Characteristic exponent : eigenvalues of J, denote $C(p)$.

- If $\mathrm{Re}(\lambda) < 0$ $\forall \lambda \in C(p)$, p is asymptotically stable.
  " >0 " unstable
- If $\mathrm{Re}(\lambda) \neq 0$ $\forall \lambda$, p is hyperbolic (Discrete system F: consider dF. Related by $\lambda \longleftrightarrow e^{\lambda}$)

If p is hyperbolic, $T_pM = E^s \oplus E^u$,
where on E^s, J is contracting and on E^u expanding.

$\Rightarrow \exists$ Immersed submfd $W^s(p)$ and $W^u(p)$, called stable and unstable mfd, which are tangent to E^s and E^u at p.

$W^s(p) = \{q : \lim_{t\to\infty} \varphi_t(q) = p\}$
$W^u(p) = \{q : \lim_{t\to-\infty} \varphi_t(q) = p\}$

Ex $F: S^2 \longrightarrow S^2$ $\Rightarrow$ Fixed pts are 0 and ∞.
$z \longmapsto 2z$

$W^u(0) = S^2 \setminus \{\infty\}$, $W^s(0) = \{0\}$.
$W^u(\infty) = \{\infty\}$, $W^s(\infty) = S^2 \setminus \{0\}$.

Suggestion
① Stable and Unstable manifolds. (Smale Ch.2)
② Return map, Characteristic multipliers (AM Ch.7.1~2)
③ Stability of Ham. system (AM Ch 8.1~3)