9/19

Goal : critical elems (fixed pt , per orb)

linearization $\downarrow$

behavior of nearby orbits

Recall $\dot{x} = X(x) \approx \underbrace{D_x X(m)}_{= J} \cdot (x - m)$ linear

fixed pt : $X(m) = 0$

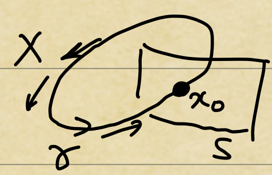
- char. exp : eig vals $\lambda_1, \cdots, \lambda_n$ of J
- char. mult : $e^{\lambda_1}, \cdots, e^{\lambda_n}$
- "elementary" or "hyperbolic" if $\mathrm{Re}(\lambda_i) \neq 0 \ \forall i$ (generic)

(Hartman–Grobman) hyperbolic $\Rightarrow$ nearby flow $\cong$ linearized flow

(e.g. $\mathrm{Re}(\lambda_i) < 0 \ . \forall i \Rightarrow$ asymp. stable)

§ Per Orb

Poincaré map γ : closed orb, period τ_0



$\exists$ ① hypersurface S s.t. $X \pitchfork S$

② $\tau : S \xrightarrow[\text{conti}]{} \mathbb{R}$ "first return time" $\tau(x_0) = \tau_0$

③ $P : S \xrightarrow[\text{diffeo}]{} S$ "Poincaré map" $P(x) = Fl_X^{\tau(x)}(x)$

P unique up to conjugation

pf : by Implicit Function thm

$(n-1)\times(n-1)$

Def char mult of γ = eig vals $c_1, \cdots, c_{n-1}$ of $D_{x_0}P$

Prop $D_{x_0} Fl_X^{\tau_0}$ has eig vals $1, c_1, \cdots, c_{n-1}$

$n \times n$

pf: $\langle X(x_0) \rangle \oplus T_{x_0} S$

$1 \qquad c_1, \cdots, c_{n-1}$

Def γ "hyperbolic" if $|c_i| \neq 1, \forall i$ (never for Ham sys)

Thm (Stable mfld Thm, Center mfld Thm)

$x \in \gamma \qquad T_x M = \underbrace{E^s}_{|c_i|<1} \oplus \underbrace{E^u}_{|c_i|>1} \oplus \underbrace{E^c}_{|c_i|=1 \,\&\, T_x\gamma}$

$\Rightarrow$ $\exists$ inv mflds $W^s_{loc}, W^u_{loc}, W^c_{loc}$ tangent to E^s, E^u, E^c at γ s.t.

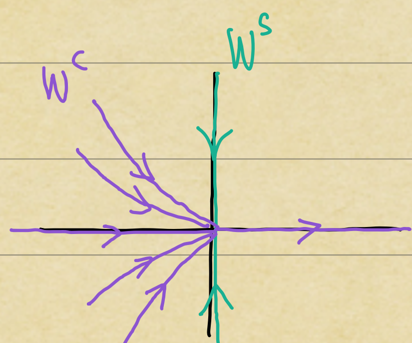
- $d(Fl_X^t(x), \gamma) \xrightarrow{t\to\infty} 0 \quad \forall x \in W^s_{loc}$
- $d(\quad " \quad) \xrightarrow{t\to-\infty} 0 \quad \forall x \in W^u_{loc}$
- W^s, W^u are unique & can be extended to "global" stable/unstable mflds

pf: construct as graphs of $y = h(x)$

+ contraction mapping thm

ex W^c may not be unique

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases}$$



$\frac{dy}{-y} = \frac{dx}{x^2} \Rightarrow y = \alpha e^{\frac{1}{x}} \qquad \alpha \in \mathbb{R}$

§ Hamiltonian Case

(M,ω) . $H \in C^\infty(M)$

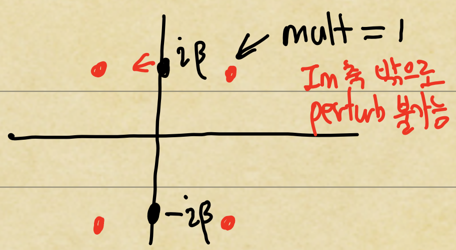
crit pt $\quad X_H(m) = 0$

linearized eqn $\quad \dot{y} = \underbrace{D_m X_H}\, y$

$\qquad\qquad\qquad\qquad = -J_0\, \mathrm{Hess}(H) \;\in sp(2n)$

$\Rightarrow$ char exp in pairs $(\lambda, -\lambda)$ or $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$

① "hyperbolic" is non-generic



② $\dim W^s = \dim W^u$, $\quad \underline{\dim W^c = 2k}$

cannot be removed by perturbation
(in general)

Ham Per. Orb

Ⓗ : Poincaré map

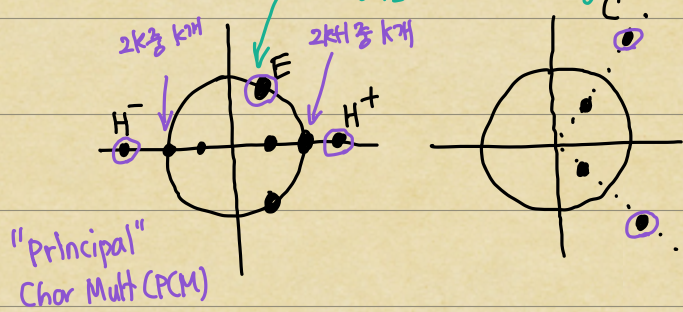
$\gamma \subset M$ closed orbit

Prop λ char mult $\Rightarrow \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ char mult

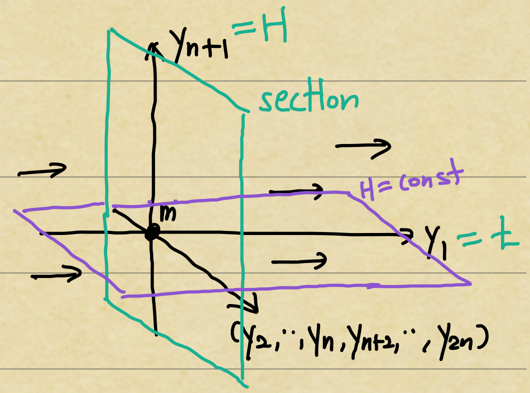
$\qquad$ 1 char mult of odd mult. ≥ 1

$(\because)$ Symp eig val thm

"unimodular" $e^{2\pi i\alpha}$
$\alpha \in [0, \frac{1}{2}]$ "transverse freq"



"Hamiltonian Flow Box" ("Power Chart")



$X_H(m) \neq 0$

$\exists$ symp coord $y_1, \cdots, y_n, y_{n+1}, \cdots, y_{2n}$

$H(y) = y_{n+1}$, motion is $\dot{y}_1 = 1$, $\dot{y}_2 = \cdots = \dot{y}_{2n} = 0$

Prop $H(m) = e$

Θ restricted to $H^{-1}(e')$ where $e' \in (e-\varepsilon, e+\varepsilon)$ is symp

Def γ : "elementary" if only one char mult $=1$ ($\Leftrightarrow$ $D_m \Theta|_{H^{-1}(e)}$ non singular)

Regular orbit cylinder

γ elementary $\Rightarrow$ $\gamma \subseteq$ cylinder of closed orbits param. by H

pf: IFT to $\Theta(x(y_{n+1}); y_{n+1}) - x(y_{n+1}) = 0$

§ Stability

$\frac{\alpha_i}{2\pi}$ "frequencies"

oscillatory case : char mult $e^{\pm i\alpha_1}, \cdots, e^{\pm i\alpha_n}$ center mfld를 고려

Q : orbital (o^+_-) stability?

가장 약한 stability이고 Lyapunov/asymptotic stability는 기대 X

Lyapunov Subcenter Thm $\Rightarrow$ 0^+-stability on subcenters

m : crit pt. frequencies $\mathbb{Q}$-lin indep ("non-resonance". "$\mathcal{H}$-elem")

$\pm i\beta \in$ char exp

$\Rightarrow$ $\exists$ 2-dim inv mfld $C_\beta \cong D^2$

$T_m C_\beta = E_{i\beta} \oplus E_{-i\beta}$

$C_\beta = \bigcup_{r \geq 0} \gamma_r$ $\leftarrow$ per orb of per $\tau_r \xrightarrow[r\to 0]{} \frac{2\pi}{\beta}$

pf: m 근방을 scaling $x \mapsto \varepsilon x$

linear system ($\varepsilon \to 0$)의 per. orb을 continue

ex Lyapunov orbits at $L_1 \sim L_3$ of CR3BP

Long/Short periodic orbits at L_4, L_5 ($0<\mu<\mu_1 \approx 0.0385$)

Moser twist stability (for 2DOF sys)

$F: U \subset \mathbb{R}^2 \to \mathbb{R}^2$ (return map)

$F(0)=0$ $DF(0)$ has eig $e^{\pm i\alpha}$ "α-twist mapping"

$\alpha \neq 0, \frac{\pi}{2}k, \frac{2\pi}{3}k$

$\Rightarrow$ $\exists$ symp chart s.t. $F(u_1+iu_2) = (u_1+iu_2)\cdot \underline{e^{i(\alpha+\beta|u|^2)}} + O(|u|^4)$

Birkhoff-Sternberg-Moser Normal form

$\alpha+\beta|u|^2$ 회전

Thm $\alpha \neq 0, \frac{\pi}{2}k, \frac{2\pi}{3}k$, $\beta \neq 0$

$\nearrow$ 0^+-stability

① $\forall$ nbd of 0, $\exists$ inv circle with no per pts

② $\forall \varepsilon>0$ $\exists \delta$ s.t. measure(inv circles in $B_\delta(0)$) $> (1-\varepsilon)2\pi\delta^2$

③ $\forall$ nbd, $\forall k$, $\exists$ fixed pt of per k